Introduction to Reachability

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Outline

- Introduction to optimal control
- Reachability as an optimal control problem
- Various shades of reachability

Goal of This Presentation

- Reachability is nothing but an optimal control/differential game problem
- Everything in reachability ultimately amounts to solving a PDE.
- Any (small enough) optimal control problem (including reachability problems) can be solved using the Level Set Toolbox.

Optimal Control

- Optimize a cost function subject to system dynamics.

- Discrete-time systems:

$$J^* = \min_{x(\cdot),u(\cdot)} \sum_{t=0}^{N-1} C(x(t),u(t),t) + l(x(N))$$

$$\text{subject to} \;\; x(t+1) = f(x(t),u(t),t); \quad x(0), N - \text{fixed}$$

- Continuous-time systems:

$$J^* = \min_{x(\cdot),u(\cdot)} \int_{t_0}^{T} C(x(t),u(t),t)dt + l(x(T))$$

$$\text{subject to} \;\; \dot{x} = f(x,u,t); \quad x(t_0), T - \text{fixed}$$

A Quick Example: Dubins Car

Terminal/Final Time

Running Cost

$$\min_{x(\cdot),u(\cdot)} \int_0^1 (x^T Q x + u^T R u)\,dt + x(1)^T Q_f x(1)$$

Initial Time

Terminal Cost

$$\text{subject to} \quad \dot{p_x} = v\cos\phi$$

$$\dot{p_y} = v\sin\phi$$

$$\dot{\phi} = \omega$$

$$x(0) = [1, 1, \tfrac{\pi}{6}]$$

Dynamics

Initial State

- Quick question: what are we trying to do to the system here?

Solving Optimal Control Problems

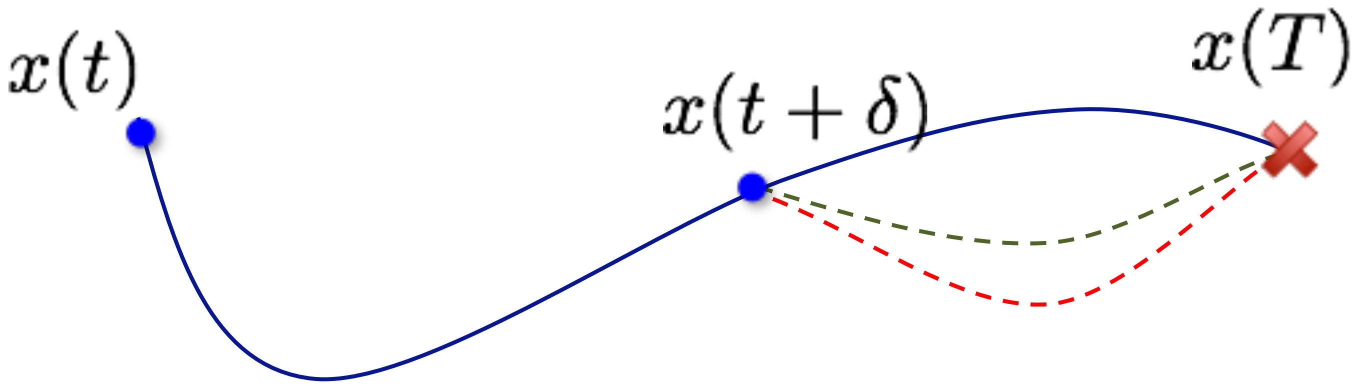
- Approach1: Calculus of Variations (CoV)
  - Takes an optimization perspective
  - Uses Lagrange multipliers to eliminate constraints
  - Derive first-order optimality conditions
  - Globally optimal solution is not guaranteed

$$J^* = \min_{x(\cdot),u(\cdot)} \int_{t_0}^{T} C(x(t), u(t), t)dt + l(x(T))$$

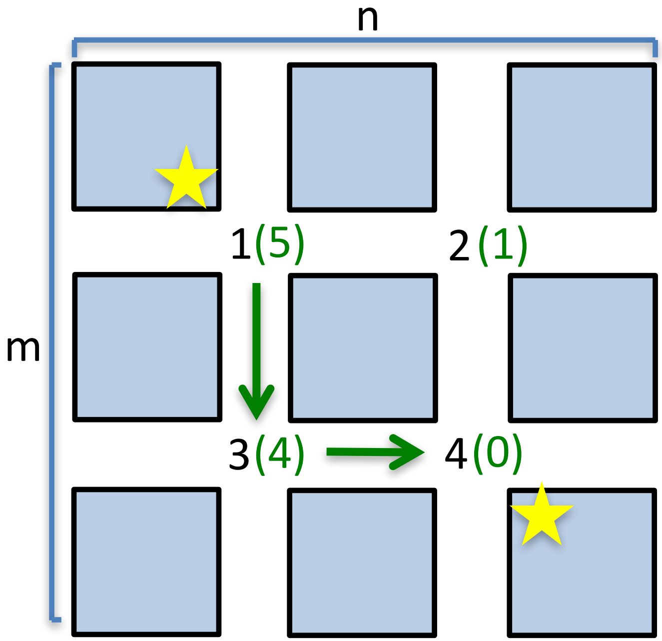
$$\text{subject to } \dot{x} = f(x, u, t); \quad x(t_0), T - \text{fixed}$$

Solving Optimal Control Problems

- Approach2: Principle of Dynamic Programming
  - Gives the globally optimal solution.
  - Principle: *The optimal state trajectory remains optimal at intermediate points in time.*



Dynamic Programming Example*



Intersection	w_{right}	w_{down}
1	5	1
2	-	1
3	4	-

Step T: $V_T(x=4)=0$

Step T-1: $V_{T-1}(2) = w_{down}(2) + V_T(4)$

$$V_{T-1}(2) = 1 + 0 = 1$$

$$V_{T-1}(3) = w_{right}(3) + V_T(4)$$

$$V_{T-1}(3) = 4 + 0 = 4$$

Step T-2:

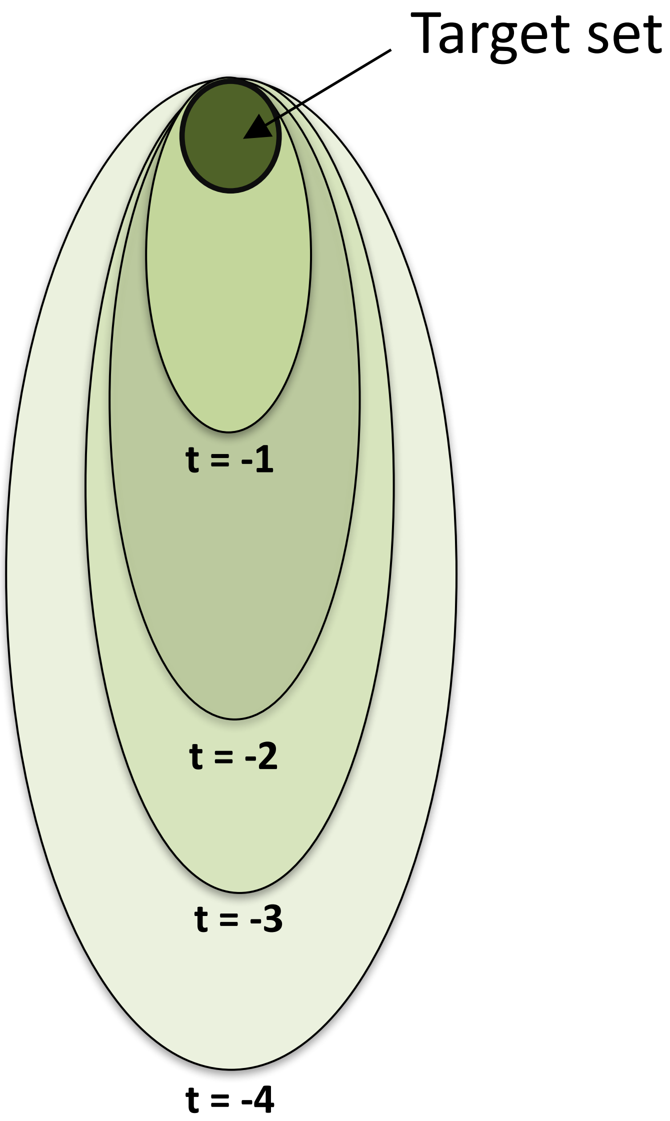
$$V_{T-2}(1) = \min\left[(w_{right}(1) + V_{T-1}(2)), (w_{down}(1) + V_{T-1}(3))\right]$$

$$V_{T-2}(1) = \min\left[(5+1), (1+4)\right] = 5$$

*Shamelessly copied from Sylvia Herbert's presentation

Magic of Dynamic Programming

- Compute the set of states that can reach the target set within 4s?
  - Assume that we can compute the set of states that can reach any other given set of states within 1s.



Principle of Dynamic Programming

- Recall our optimal control problem:

$$\text{Minimize} \quad J(x,t) = \int_t^T C\big(x(t),u(t)\big)dt + l\big(x(T)\big)$$

$$\text{Subject to } \dot{x} = f(x,u,t)$$

- Define cost from (t,x)

$$V(x(t),t) = \min_{u(\cdot)} \left[\int_t^T C\big(x(t),u(t)\big)dt + l\big(x(T)\big)\right]$$

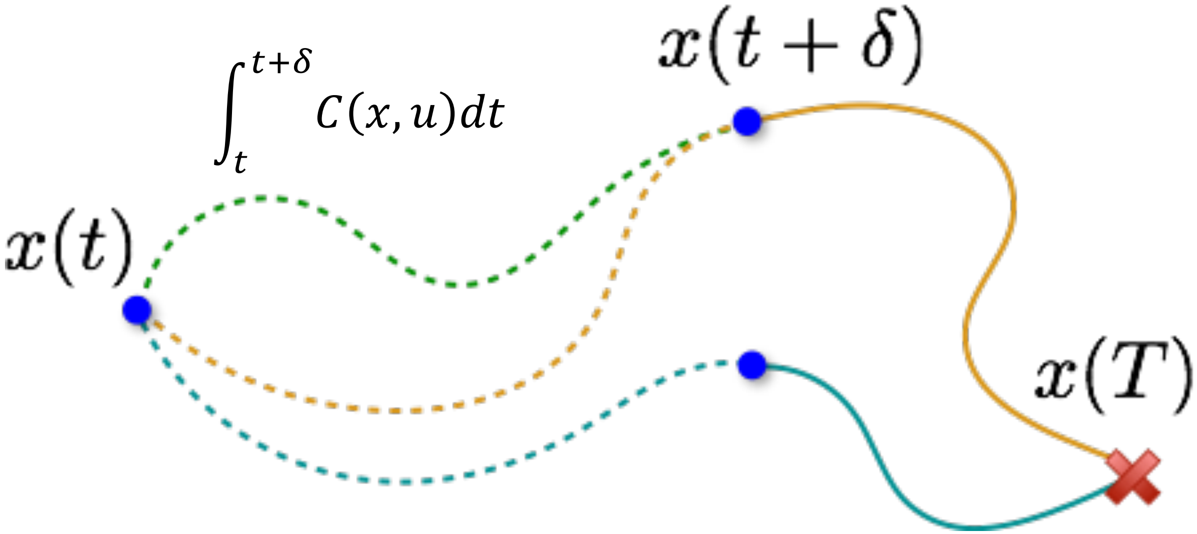
- We are interested in finding cost from state x and time 0

$$V(x(0),0) = \min_{u(\cdot)} \left[\int_0^T C\big(x(t),u(t)\big)dt + l\big(x(T)\big)\right]$$

Principle of Dynamic Programming

- Dynamic programming principle implies that:

$$V(x(t),t) = \min_u \left[V(x(t+\delta), t+\delta) + \int_t^{t+\delta} C(x,u)dt \right]$$



Principle of Dynamic Programming

$$V(x(t),t) = \min_{u}\left[V(x(t+\delta),t+\delta) + \int_{t}^{t+\delta} C(x,u)dt\right]$$

$$V(x(t),t) + \frac{dV}{dt}(x(t),t)\delta + \nabla V(x(t),t)\cdot\frac{dx}{dt}\delta \qquad C(x,u)\delta$$

Taylor Expansion

Approximation

$$V(x(t),t) = \min_{u(t)}\left[V(x(t),t) + \frac{dV}{dt}(x(t),t)\delta + \nabla V(x(t),t)\cdot\frac{dx}{dt}\delta + C(x,u)\delta\right]$$

Principle of Dynamic Programming

$$V(x(t),t) = \min_{u(t)} \left[V(x(t),t) + \frac{dV}{dt}(x(t),t)\delta + \nabla V(x(t),t) \cdot \frac{dx}{dt}\delta + C(x,u)\delta \right]$$

$$V(x(t),t) = V(x(t),t) + \min_{u(t)} \left[\frac{dV}{dt}(x(t),t)\delta + \nabla V(x(t),t) \cdot \frac{dx}{dt}\delta + C(x,u)\delta \right]$$

$$\cancel{V(x(t),t)} = \cancel{V(x(t),t)} + \min_{u(t)} \left[\frac{dV}{dt}(x(t),t)\delta + \nabla V(x(t),t) \cdot \frac{dx}{dt}\delta + C(x,u)\delta \right]$$

$$0 = \min_{u(t)} \left[\frac{dV}{dt}(x(t),t)\delta + \nabla V(x(t),t) \cdot \frac{dx}{dt}\delta + C(x,u)\delta \right]$$

$$0 = \min_{u(t)} \left[\frac{dV}{dt}(x(t),t) + \nabla V(x(t),t) \cdot \frac{dx}{dt} + C(x,u) \right]$$

Principle of Dynamic Programming

$$0 = \min_{u(t)} \left[\frac{dV}{dt}(x(t), t) + \nabla V(x(t), t) \cdot \frac{dx}{dt} + C(x, u) \right]$$

$$\frac{dV}{dt} + \min_{u} \{ \nabla V(x(t), t) \cdot f(x, u) + C(x, u) \} = 0$$

Hamilton-Jacobi Bellman PDE

$$V(x(T), T) = l\big(x(T)\big)$$

Final Value of PDE

Hamilton-Jacobi Bellman(HJB) PDE

Problem:

Minimize $J(x,t) = \int_t^T C\big(x(t),u(t)\big)dt + l\big(x(T)\big)$

Subject to $\dot{x} = f(x,u,t)$

Solution

A final-value PDE

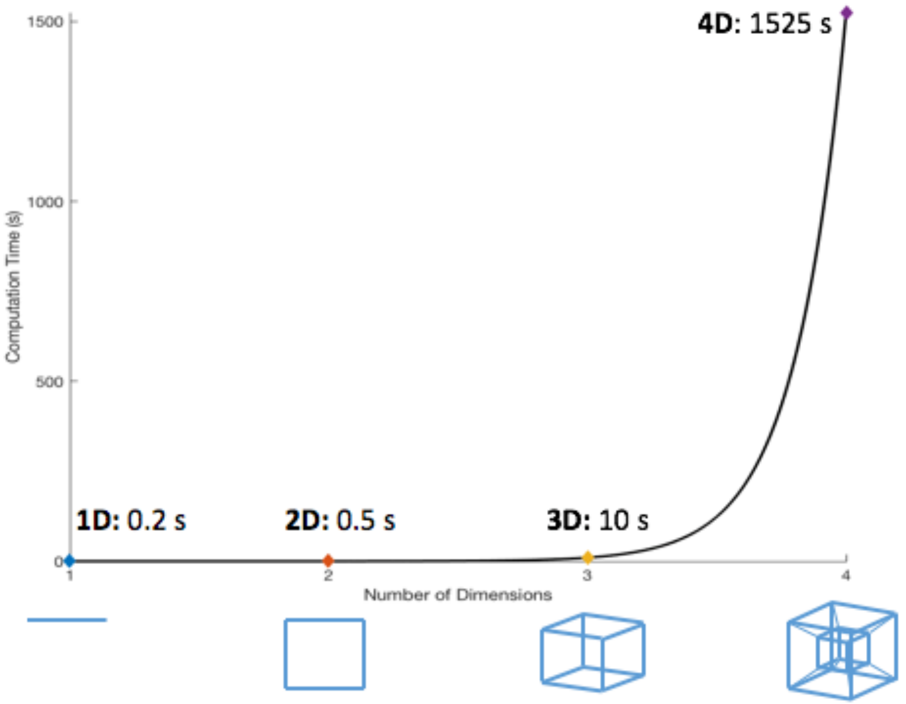
$$\frac{dV}{dt} + \min_u\{\nabla V(x(t),t)\cdot f(x,u) + C(x,u)\} = 0$$

$$V(x(T),T) = l\big(x(T)\big)$$

- What is $V(x(t),t)$?
- Where is my optimal control?

Curse of Dimensionality

- Wow! One single PDE for any optimal control problem.
  - What is the problem here?



Level Set Toolbox

- Developed by Prof. Ian Mitchell during his PhD

- Can solve any initial-value PDE of the form:

$$\frac{dV}{dt} + H^*(x, \nabla V(x(t), t), t) = 0$$

$$V(x(0), 0) = l\big(x(0)\big)$$

But optimal control problem is a final value PDE ….☹

$$\frac{dV}{dt} + \min_u\{\nabla V(x(t), t) \cdot f(x, u) + C(x, u)\} = 0$$

$$V(x(T), T) = l\big(x(T)\big)$$

Level Set Toolbox

- Good news: They are interchangeable!

$$\frac{dV}{dt} + H^*(x, \nabla V(x(t), t), t) = 0$$

$$V(x(0), 0) = l\big(x(0)\big)$$

$$=$$

$$W(x, T - t) = V(x, t)$$

$$\frac{dW}{dt} - H^*(x, \nabla W(x(t), t), t) = 0$$

$$W(x(T), T) = l\big(x(T)\big)$$

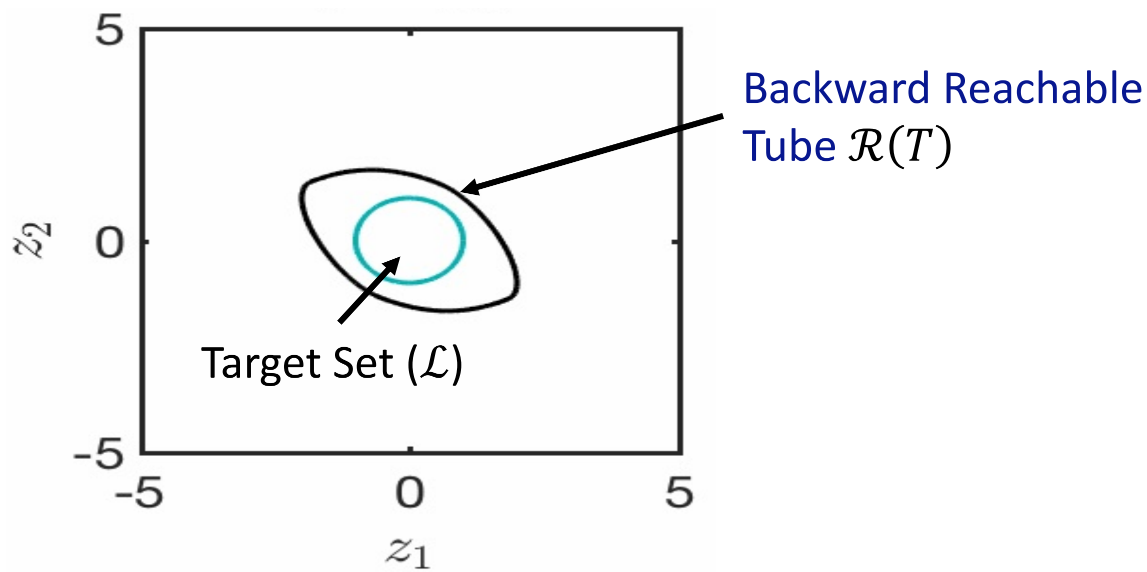
- So we can solve any optimal control problem with the toolbox!

Optimal Control: Quick Recap

- Optimal control problem:
  - Optimize a cost function subject to system dynamics.
- Two approaches:
  - Calculus of Variations:
    - Gives local solutions, but faster to compute.
  - Dynamic Programming:
    - Gives global solution
    - A Final-value PDE needs to be solved
      - May not even have a classical solution!
      - Curse of dimensionality!
      - Can be solved using Level Set Toolbox for low dimensional problems.

Reachability

- Problem: Find the set of all states that can reach a given set of states $\mathcal{L}$ within a time duration of T.



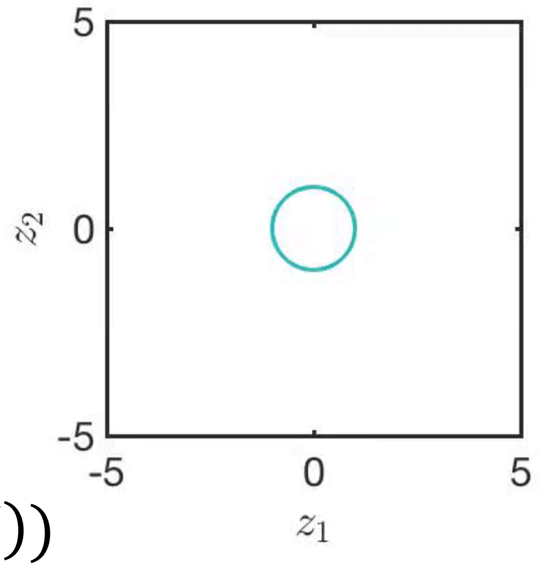
$$\mathcal{R}(T) = \{\mathrm{x}_0 : \exists u, s.t.\ x(\cdot)\ \textit{satisfies}\ \dot{x} = f(x,u), x(0) = x_0; \exists t \in [0,T], s.t.\ x(t) \in \mathcal{L}\}$$

- Any thoughts?

Reachability

- Hint 1: Define a function $l(x)$ such that,

$$\mathcal{L} = \{x: l(x) \leq 0\}$$



Now consider the problem,

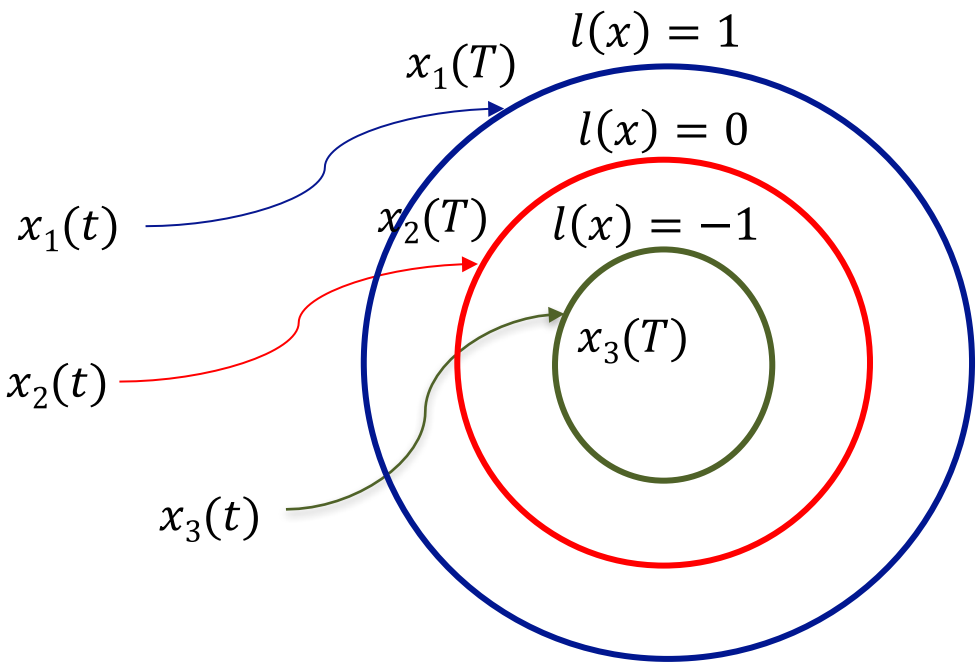
$$V(x(t), t) = \min_u \; l(x(T))$$

$$\text{Subject to } \dot{x} = f(x, u, t)$$

- What does $V(x(t), t)$ represent?

Reachability

- What are $V(x(t), t)$ for each of the following trajectories?



$$V(x(t), t) = \min_u \; l(x(T))$$

Reachability

- So what does $V(x(t), t)$ represent?
  - The value of $l(x)$ that we will reach at time T
- How do I answer my original question?

$$x(0) \in \mathcal{R}(T) \Leftrightarrow V(x(0), 0) \leq 0$$

$$x(0) \notin \mathcal{R}(T) \Leftrightarrow V(x(0), 0) > 0$$

Reachability

- So reachability is nothing but an optimal control problem.

$$\min_{u} \; l(x(T))$$

$$\text{Subject to } \dot{x} = f(x,u,t)$$

$$\mathcal{L} = \{x : l(x) \leq 0\}$$

- What is the corresponding PDE?

Co-state

Value Function

Hamiltonian

$$\frac{dV}{dt} + H^{*}(x, \nabla V(x(t),t), t) = 0$$

$$V(x(T),T) = l\big(x(T)\big)$$

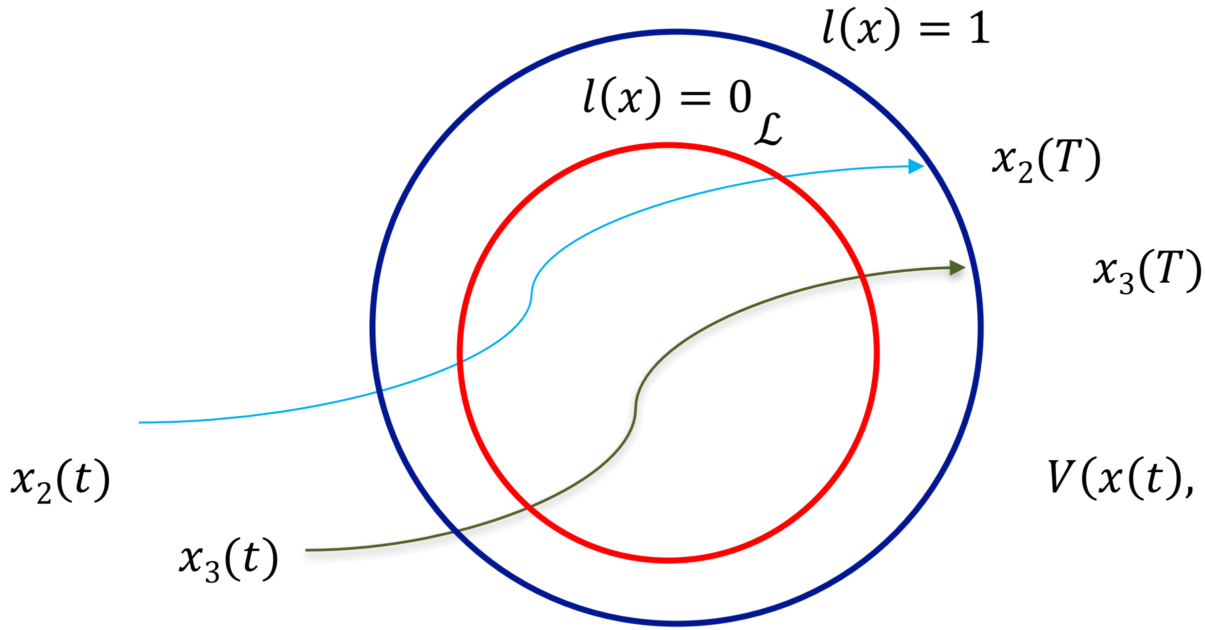
$$H^{*} = \min_{u}\{\nabla V(x(t),t) \cdot f(x,u,t)\}$$

- How to get "target-reaching” control?

$$u^{*} = \operatorname*{argmin}_{u}\{\nabla V(x(t),t) \cdot f(x,u,t)\}$$

But hold on…

- What are $V(x(t), t)$ for each of the following trajectories?



$$V(x(t), t) = \min_u \, l(x(T))$$

- What they should be?

$$\mathcal{R}(T) = \{x_0 \colon \exists u, s.t.\ x(\cdot)\ \textit{satisfies}\ \dot{x} = f(x, u), x(0) = x_0; \exists t \in [0, T], s.t.\ x(t) \in \mathcal{L}\}$$

$$x(0) \in \mathcal{R}(T) \Leftrightarrow V(x(0), 0) \le 0$$

What's going on?

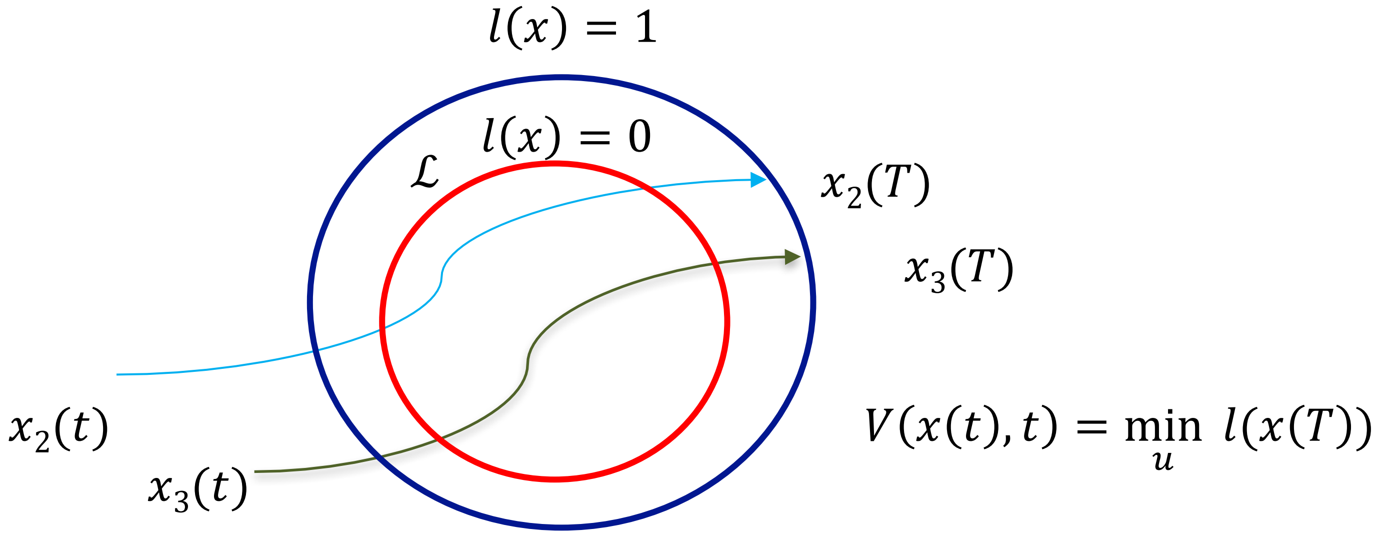
- Need to account for the fact that trajectories can reach the target but then leave it.

$$\min_u \left(\min_{t\in[0,T]} l(x(t)) \right)$$

Subject to $\dot{x} = f(x,u,t)$

$$\mathcal{L} = \{x: l(x) \le 0\}$$

- Does this fix the issue?



Freezing the Trajectories in the Target

- Need to account for the fact that trajectories can reach the target but then escape it.

$$\min_u \left(\min_{t\in[0,T]} l(x(t)) \right)$$

Subject to $\dot{x} = f(x,u,t)$

$$\mathcal{L} = \{x: l(x) \leq 0\}$$

- What is the corresponding PDE?

$$\frac{dV}{dt} + \min\{0, H^*(x, \nabla V(x(t),t),t)\} = 0$$

$$V(x(T),T) = l\big(x(T)\big)$$

$$H^* = \min_u\{\nabla V(x(t),t) \cdot f(x,u,t)\}$$

Reachability: Reachable Sets vs Tubes

- Backward Reachable Set (BRS): the set of all states that can reach a target set of states $\mathcal{L}$ **exactly** at time T.

$$\mathcal{R}'(T) = \{\mathrm{x}_0 \colon \exists u, s.t.\ x(\cdot)\ satisfies\ \dot{x} = f(x,u), x(0) = x_0; x(T) \in \mathcal{L}\}$$

$$\min_u \ l(x(T))$$
Subject to $\dot{x} = f(x,u,t)$
$$\mathcal{L} = \{x \colon l(x) \le 0\}$$

$$\frac{dV}{dt} + H^*(x, \nabla V(x(t),t),t) = 0$$
$$V(x(T),T) = l(x(T))$$
$$H^* = \min_u \{\nabla V(x(t),t) \cdot f(x,u,t)\}$$

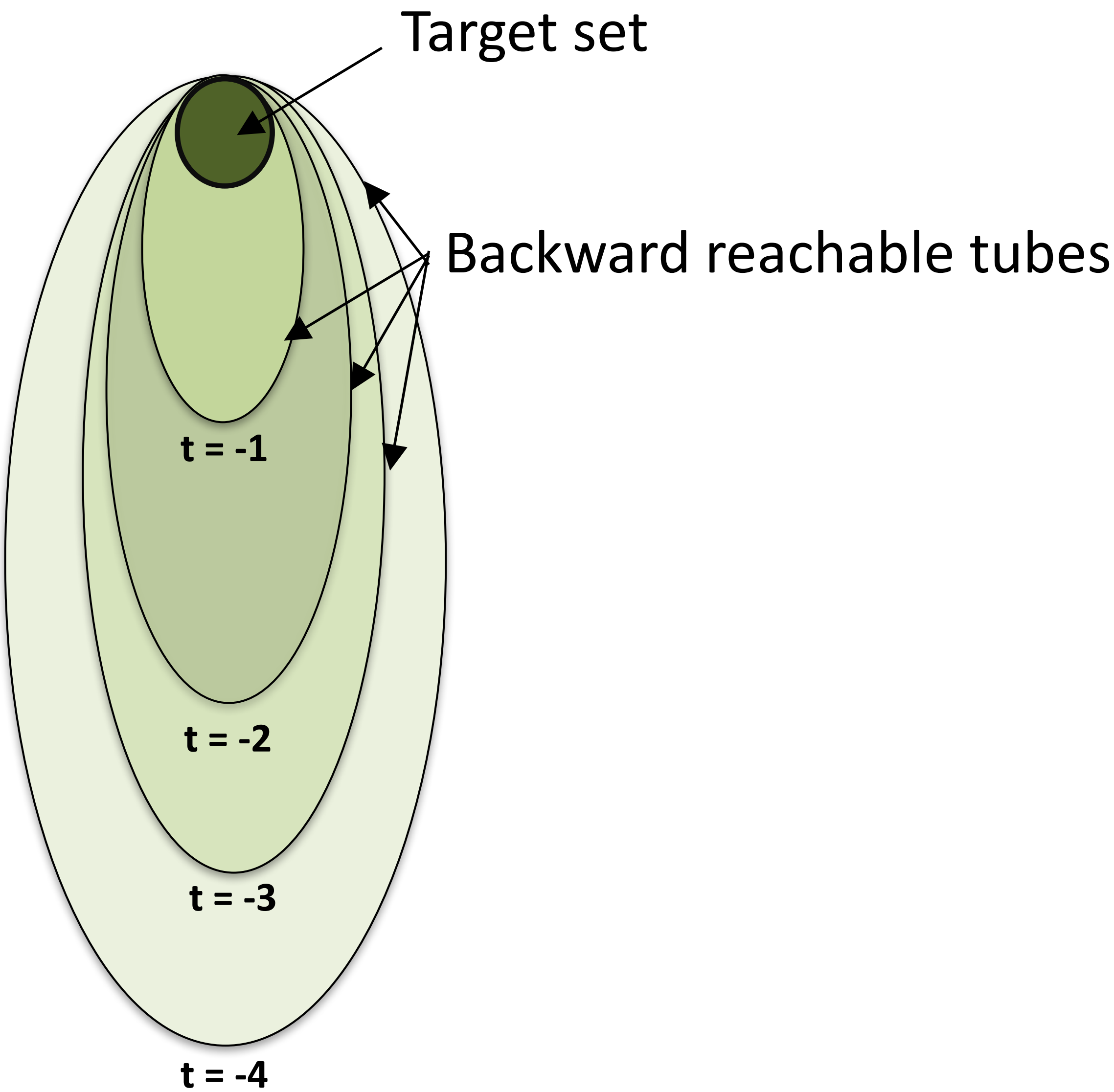
- Backward Reachable Tube (BRT): the set of all states that can reach a target set of states $\mathcal{L}$ **within** a duration of time T.

$$\mathcal{R}(T) = \{\mathrm{x}_0 \colon \exists u, s.t.\ x(\cdot)\ satisfies\ \dot{x} = f(x,u), x(0) = x_0; \exists t \in [0,T], s.t.\ x(t) \in \mathcal{L}\}$$

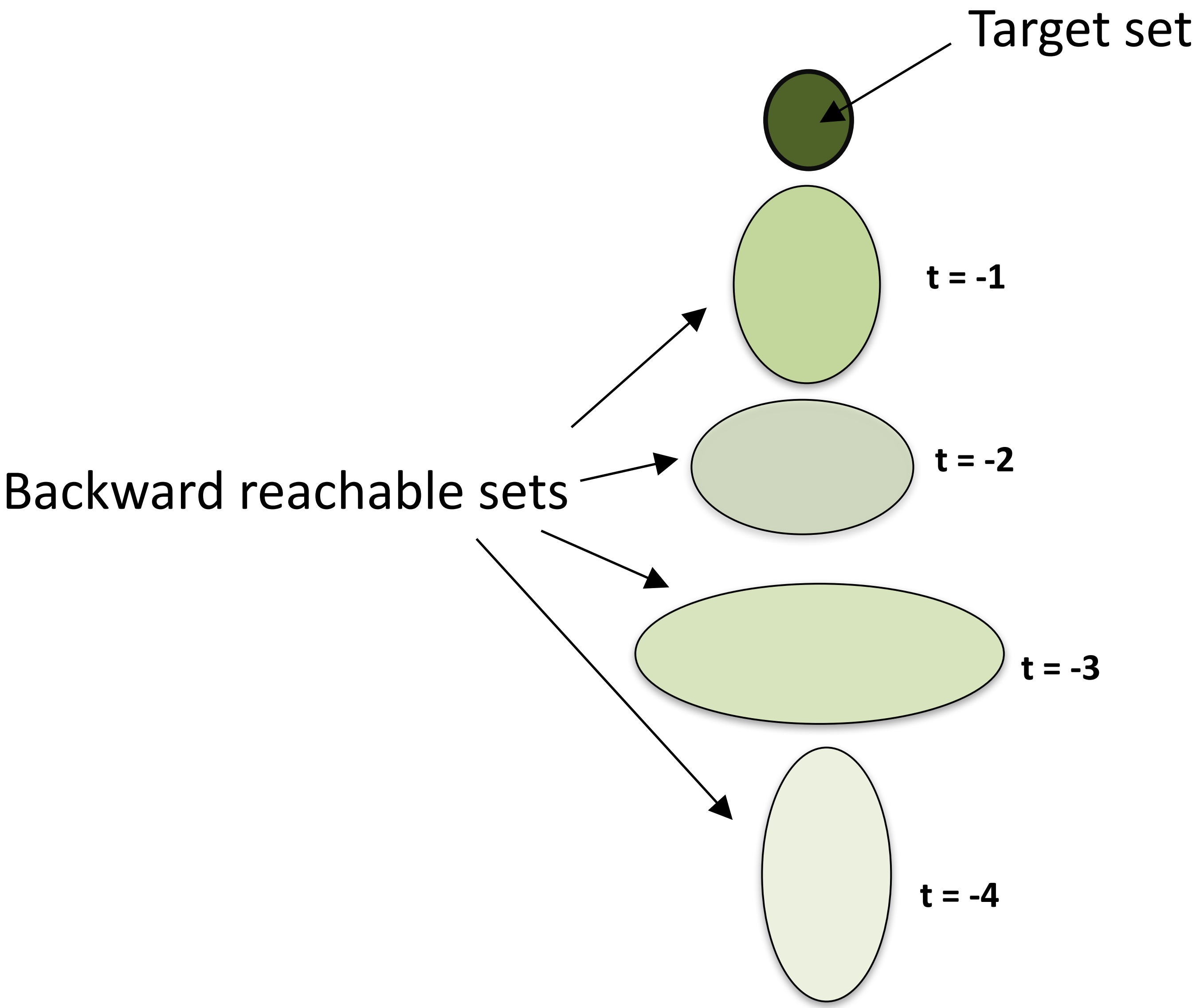
$$\min_u \left(\min_{t\in[0,T]} l(x(t))\right)$$
Subject to $\dot{x} = f(x,u,t)$
$$\mathcal{L} = \{x \colon l(x) \le 0\}$$

$$\frac{dV}{dt} + \min\{0, H^*(x, \nabla V(x(t),t),t)\} = 0$$
$$V(x(T),T) = l(x(T))$$
$$H^* = \min_u \{\nabla V(x(t),t) \cdot f(x,u,t)\}$$

Backward Reachable Tube: Example



Backward Reachable Set: Example



Computing Backward Reachable Tube

1. Define target set $\mathcal{L}$ for the system to reach within a given time horizon:

2. Define implicit level set function for final time $l(z)$, $\mathcal{L} = \{z: l(z) \leq 0\}$

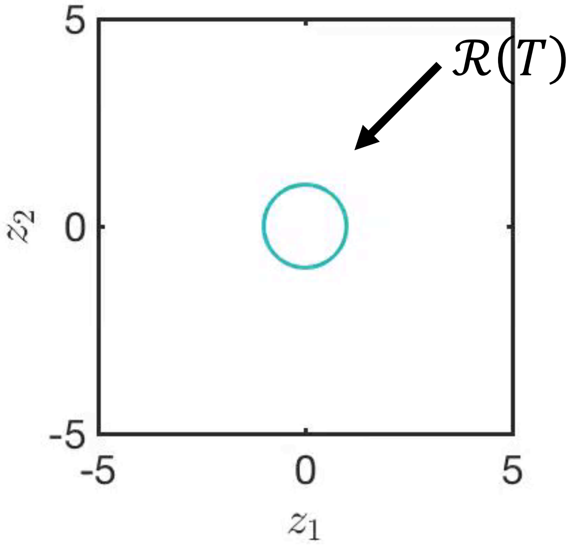
3. Find an appropriate value function $V(z(t), t)$

$$\frac{dV}{dt} + \min\{0, H^*(x, \nabla V(x(t), t), t)\} = 0$$

$$V(x(T), T) = l\big(x(T)\big)$$

$$H^* = \min_u\{\nabla V(x(t), t) \cdot f(x, u, t)\}$$

4. Retrieve zero sub-level of level set function at initial time



Reachability: Key Takeaways

- Reachability is just an optimal control problem.
- Backward Reachable Set (BRS) vs Backward Reachable Tube (BRT)
- Both can be computed using the Level Set Toolbox.
- Suffers from the curse of dimensionality

Introducing the Disturbance

- Suppose our dynamics were: $\dot{x} = f(x, u, d)$
  - u – control, d – disturbance

- And cost were: $J(x,t) = \int_t^T C\big(x(t), u(t), d(t)\big)dt + l\big(x(T)\big)$

- Now, we want to solve the following *differential* game:

$$V(x(t), t) = \min_{u(\cdot)} \max_{d(\cdot)} \left[\int_t^T C\big(x(t), u(t), d(t)\big)dt + l\big(x(T)\big)\right]$$

- A similar PDE can be derived in this case (called HJI PDE)

Optimal Control vs Differential Game

	Static	Evolving Over Time
One agent (input)	Optimization	Optimal Control
Multiple agents (input and disturbance)	Game Theory	Differential Games

Hamilton-Jacobi Isaacs(HJI) PDE

Problem:

Minimize $J(x,t) = \int_t^T C(x(t),u(t),d(t))dt + l\big(x(T)\big)$

Subject to $\dot{x} = f(x,u,d,t)$

Solution

A final-value PDE

$$\frac{dV}{dt} + \min_u \max_d \{\nabla V(x(t),t) \cdot f(x,u,d) + C(x,u,d)\} = 0$$

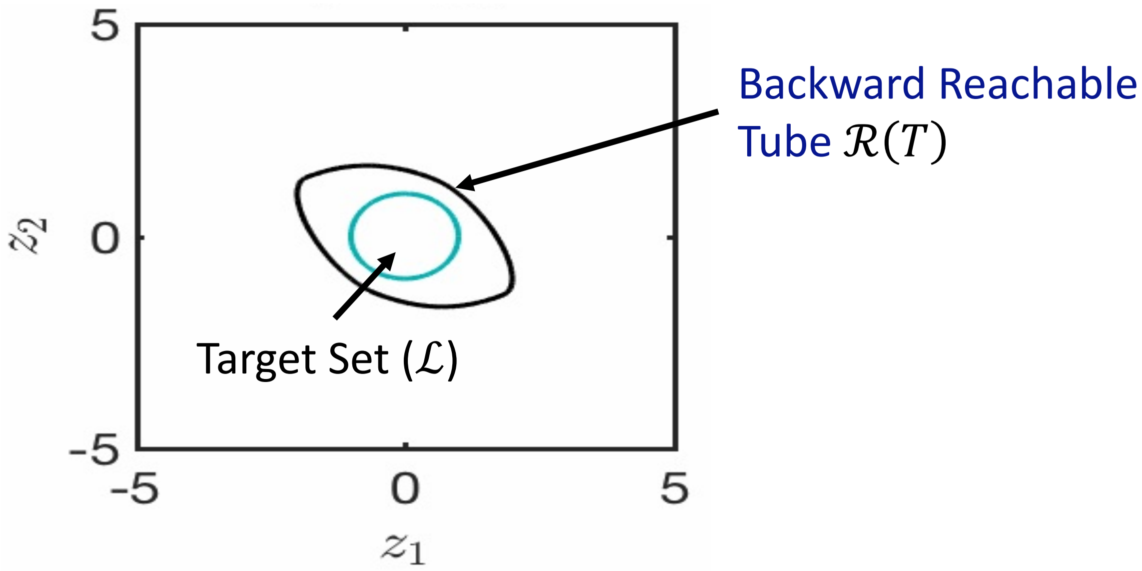
$$V(x(T),T) = l\big(x(T)\big)$$

- What is $V(x(t),t)$?

- Where is my optimal control?

Reachability With Disturbance

- Problem: Find the set of all states that can *reach* a given set of states $\mathcal{L}$ *despite the disturbance* within a time duration of T.



$$\mathcal{R}(T) = \{x_0 \colon \exists u, s.t.\ \forall d, x(\cdot) \text{ satisfies } \dot{x} = f(x, u, d), x(0) = x_0; \exists t \in [0, T], s.t.\ x(t) \in \mathcal{L}\}$$

- What does this definition mean?
- How to solve this problem?

Reachability With Disturbance

- We can again formulate it as a differential game

$$\min_u \max_d \min_{t\in[0,T]} l(x(t))$$

Subject to $\dot{x} = f(x, u, d, t)$

$$\mathcal{L} = \{x: l(x) \le 0\}$$

- What is the corresponding PDE?

$$\frac{dV}{dt} + \min\{0, H^*(x, \nabla V(x(t), t), t)\} = 0$$

$$V(x(T), T) = l\big(x(T)\big)$$

$$H^* = \min_u \max_d \{\nabla V(x(t), t) \cdot f(x, u, d, t)\}$$

- How to get "target-reaching” control?

$$u^* = \underset{u}{\operatorname{argmin}} \max_d \{\nabla V(x(t), t) \cdot f(x, u, d, t)\}$$

Reachability With Disturbance: Key Takeaways

- Very similar to classic reachability; just an extra max
- Backward Reachable Set (BRS) vs Backward Reachable Tube (BRT)
- Both can be computed using the Level Set Toolbox.
- Suffers from the curse of dimensionality

Reachability With Disturbance: Quick Trivia

- Problem: Find the set of all states that can *reach* a given set of states $\mathcal{L}$ *for some* disturbance within a time duration of T.

$$\mathcal{R}(T) = \{\mathrm{x}_0 \colon \exists u, \exists d, s.t.\ x(\cdot)\ \textit{satisfies}\ \dot{x} = f(x,u,d), x(0) = x_0; \exists t \in [0,T], s.t.\ x(t) \in \mathcal{L}\}$$

- What does this definition mean?

- How to solve this problem?

Reachability With Disturbance: Trivia Solution

- Disturbance is like control here

$$\min_u \min_d \min_{t\in[0,T]} l(x(t))$$

$$\text{Subject to } \dot{x} = f(x, u, d, t)$$

$$\mathcal{L} = \{x: l(x) \leq 0\}$$

- What is the corresponding PDE?

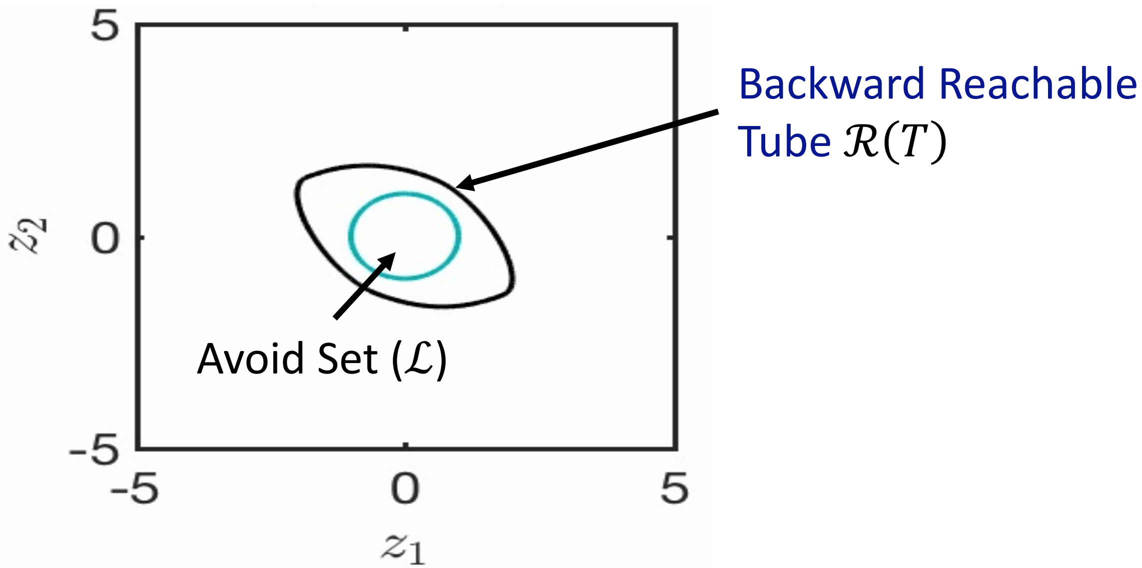
$$\frac{dV}{dt} + \min\{0, H^*(x, \nabla V(x(t), t), t)\} = 0$$

$$V(x(T), T) = l\big(x(T)\big)$$

$$H^* = \min_u \min_d \{\nabla V(x(t), t) \cdot f(x, u, d, t)\}$$

Shades of Reachability: Avoid Set

- Problem: Find the set of all states that can *avoid* a given set of states $\mathcal{L}$ *despite the disturbance* for a time duration of T.



$$\mathcal{R}(T) = \{x_0 : \exists d, s.t.\ \forall u, x(\cdot) \textit{ satisfies } \dot{x} = f(x,u,d), x(0) = x_0; \exists t \in [0,T], s.t.\ x(t) \in \mathcal{L}\}$$

- What does this definition mean?
- How to solve this problem?

Reachability: Avoid Set

- Simply exchange the role of input and disturbance

$$\max_u \min_d \min_{t\in[0,T]} l(x(t))$$

Subject to $\dot{x} = f(x,u,d,t)$

$$\mathcal{L} = \{x: l(x) \le 0\}$$

- What is the corresponding PDE?

$$\frac{dV}{dt} + \min\{0, H^*(x, \nabla V(x(t),t),t)\} = 0$$

$$V(x(T),T) = l\big(x(T)\big)$$

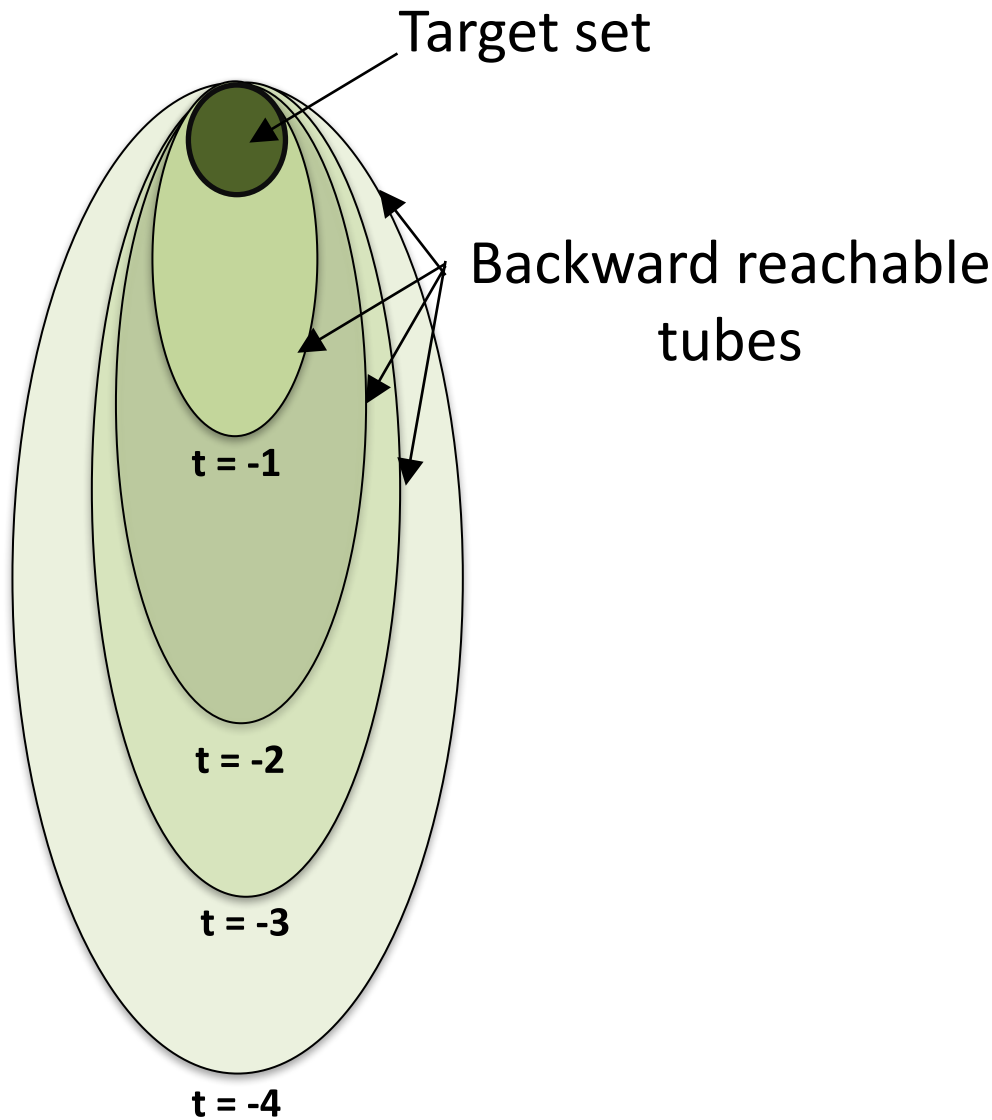
$$H^* = \max_u \min_d \{\nabla V(x(t),t) \cdot f(x,u,d,t)\}$$

- How to get "target-avoiding” control?

$$u^* = \operatorname*{argmax}_u \min_d \{\nabla V(x(t),t) \cdot f(x,u,d,t)\}$$

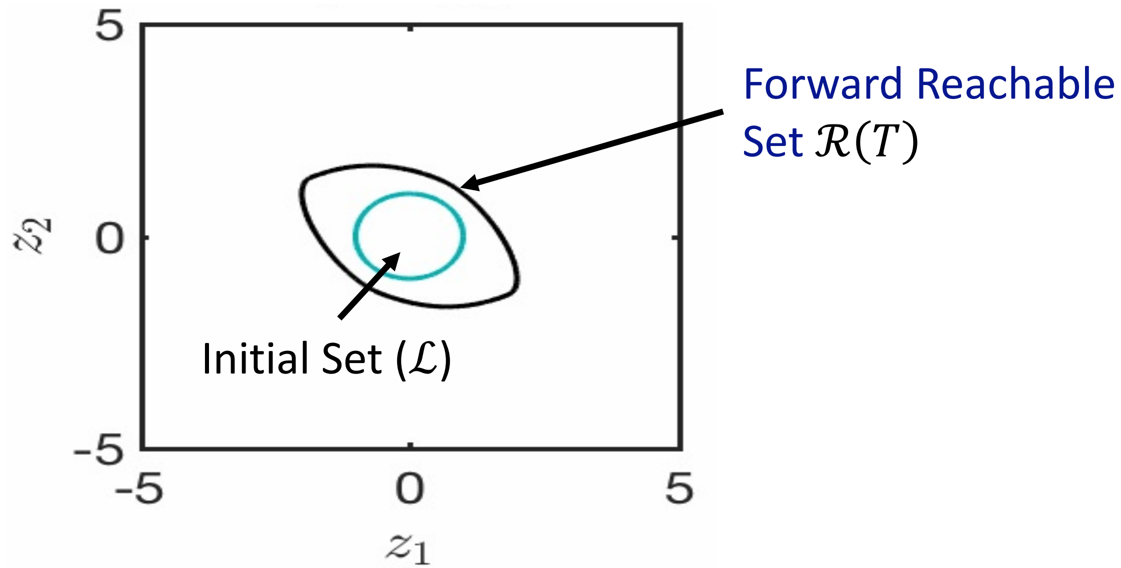
Avoid Set: Example

- What do the green sets represent here?



Shades of Reachability: Forward Reachable Set

- Problem: Find the set of all states that I can reach *from* a given set of states $\mathcal{L}$ at time T.



$$F(T) = \{x_T: \exists u, s.t.\ x(\cdot)\ satisfies\ \dot{x} = f(x,u), x(0) \in \mathcal{L};\ x(T) = x_T\}$$

- What does this definition mean?
- How to solve this problem?

Forward Reachable Set

- Define a function $l(x)$ such that,

$$\mathcal{L} = \{x: l(x) \leq 0\}$$

Now consider the problem,

$$V(x(t), t) = \max_{u} \; l(x(T))$$

$$\text{Subject to } \dot{x} = f(x, u, t)$$

- Will this work?

Forward Reachable Set

- Again it is an optimal control problem.

$$\max_u \; l(x(T))$$

$$\text{Subject to } \dot{x} = f(x,u,t)$$

$$\mathcal{L} = \{x : l(x) \leq 0\}$$

- What is the corresponding PDE?

An Initial Value PDE

$$\frac{dV}{dt} + H^*(x, \nabla V(x(t),t),t) = 0$$

$$V(x(0),0) = l\big(x(0)\big)$$

$$H^* = \max_u \{\nabla V(x(t),t) \cdot f(x,u,t)\}$$

Forward Reachable Set Trivia

- Problem: Find the set of all states that I can reach *from* a given set of states $\mathcal{L}$ *despite the disturbance* at time T.

$$F(T) = \{\mathrm{x}_T : \exists u, s.t.\ \forall d, x(\cdot)\ \textit{satisfies}\ \dot{x} = f(x,u,d), x(0) \in \mathcal{L};\ x(T) = \mathrm{x}_T\}$$

- Solve the following PDE:

$$\frac{dV}{dt} + H^*(x, \nabla V(x(t),t),t) = 0$$

$$V(x(0),0) = l\big(x(0)\big)$$

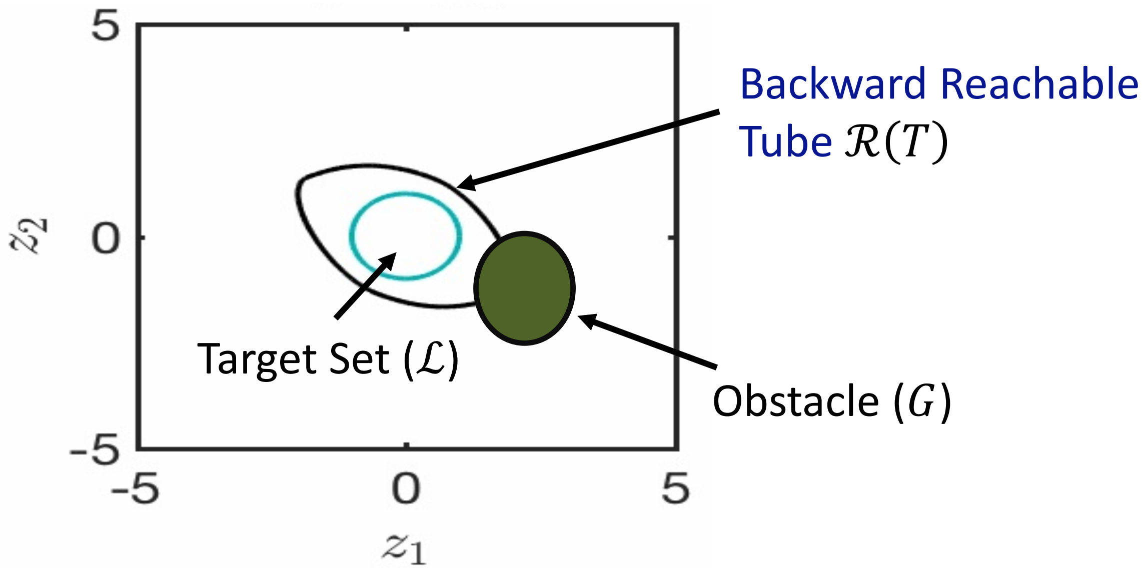
$$H^* = \max_u \min_d \{\nabla V(x(t),t) \cdot f(x,u,t)\}$$

Forward Reachable Set: Key Takeaways

- Gives an initial value PDE
- Forward Reachable Set (FRS) vs Forward Reachable Tube (FRT)
- Both can be computed using the Level Set Toolbox.
- Suffers from the curse of dimensionality

Shades of Reachability: Obstacles

- Problem: Find the set of all states that can *reach* a given set of states $\mathcal{L}$ *without hitting the obstacle* (G) *despite the disturbance* within a time duration of T.



$$\mathcal{R}(T) = \left\{ \begin{array}{l} x_0: \exists u, s.t.\ \forall d, x(\cdot) \text{ satisfies } \dot{x} = f(x,u,d), x(0) = x_0; \\ \quad \forall t \in [0,T], x(t) \notin G \text{ and } \exists t \in [0,T], s.t.\ x(t) \in \mathcal{L} \end{array} \right\}$$

- What does this definition mean?

Reachability With Obstacles

- Again can be formulated as a differential game

- Value function can be shown to satisfy the following equation:

$$\max\left\{ \frac{dV}{dt} + \min\{0, H*(x, \nabla V(x(t),t),t)\}, g(x) - V(x,t)\right\} = 0$$

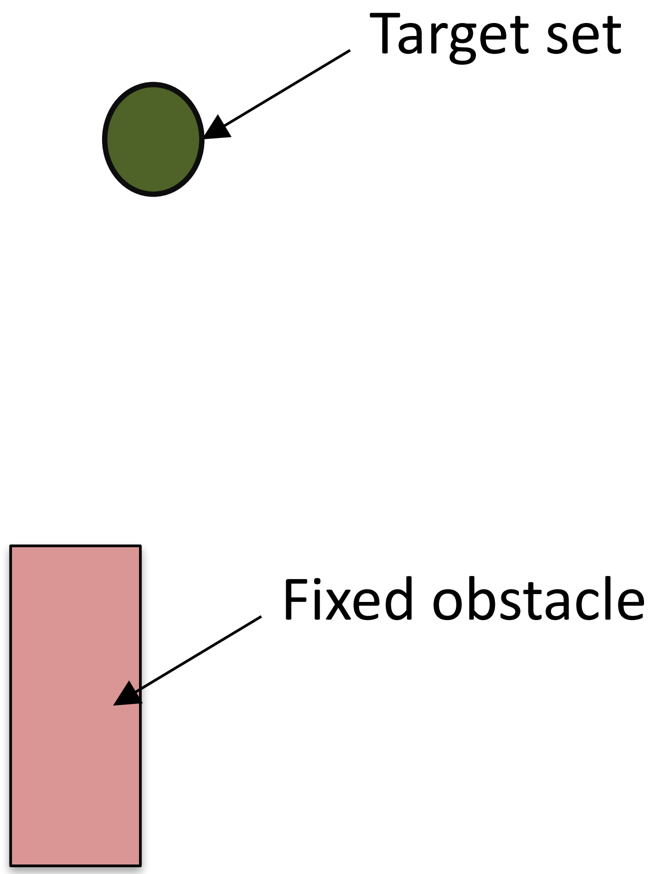
$$V(x(T),T) = \max\{l(x,T), g(x,T)\}$$

$$H^* = \min_u \max_d \{\nabla V(x(t),t) \cdot f(x,u,d,t)\}$$

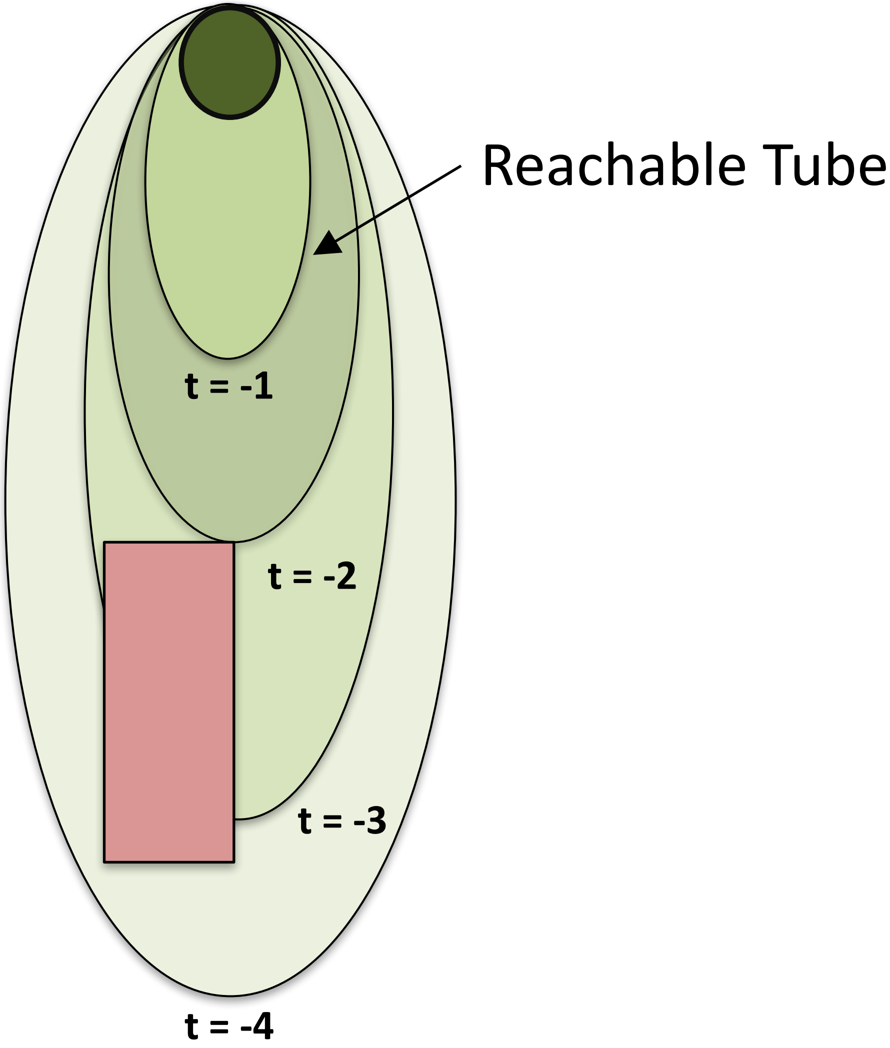
- How to get "target-reaching” control without hitting the obstacle?

$$u^* = \underset{u}{\mathrm{argmin}} \max_d \{\nabla V(x(t),t) \cdot f(x,u,d,t)\}$$

Reachability With Obstacles: Example



Reachability With Obstacles: Example



Shades of Reachability: Key Takeaways

- Everything in reachability ultimately amounts to solving a PDE.
- Different min-max combinations appear in the PDE based on what control and disturbance are trying to do.
- Min with zero appears in the PDE depending on whether a set or a tube is being computed.
- Initial or final value PDE is solved based on whether a FRS or a BRS is being computed
- Obstacles can be considered
- Any combination above can be computed using the Level Set Toolbox.

Reachability: Final Remarks

- Reachability theory has much more rigorous mathematical foundation.

- Time-varying targets/obstacles can also be treated very easily.

Thank You!

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