Project Report

EE 227BT

Convex Relaxation of Optimal Power Flow Problem

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Introduction

Optimal power flow problem (OPF) is fundamental in power system operations as it underlies many applications such as economic dispatch, unit commitment, state estimation, stability and reliability assessment, volt/var control, demand response, etc. In general, OPF refers to any mathematical program that seeks to minimize a certain function, such as total power loss, generation cost or user disutility, subject to the Kirchhoff's laws for a power network, as well as capacity and stability constraints on bus voltages and generators. Even when the objective function is convex, the power flow constraints (given by the Kirchhoff's laws) are generally non-convex and hence solving an OPF is NP-hard. To overcome this problem, a large number of relaxations as well as approximations have been proposed. A popular approximation, for example, is a linear program, called DC OPF, obtained through the linearization of the power flow equations. Similarly, SOCP relaxation is among the most tractable convex relaxations of the power flow equations.

In this project, we will study the SOCP relaxation of power flow equations in distribution networks. We will start with introducing the model for distribution networks, which have a tree structure and hence also called radial networks, derive its power flow equations using the Kirchhoff's laws, introduce other stability and capacity constraints, and define the OPF problem. We will explain why the resultant problem is non-convex, relax the feasible set of this problem, and show that the resultant problem is an SOCP. We next study the tightness of this relaxation, present the conditions under which the relxation is exact, and explain how to recover a solution of the original OPF problem. A linear approximation of power flow equations, called DC OPF, has also been presented, which is widely famous in the power system research community due to its simplicity. The main goal of this project is to compare the three approaches: exact solution of non-convex OPF (computed using sequential convex programming, wherever possible), the SOCP relaxation, and the linear approximation. In particular, we simulated the three approaches on a real network to compare the optimal costs and the line voltages. We found that for most practical systems SOCP will likely provide the exact solution to OPF and hence can be used as a tractable tool to solve OPF problem. Linear approximation on the other hand can be very crude, or even infeasible even when the original OPF is feasible, depending on the load conditions, and hence is not a practical alternative.

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Chapter 1

Introduction to power systems

An electric power system is a network of electrical components used to supply, transfer and use electric power. An example of an electric power system is the network that supplies a region's homes and industry with powerfor sizable regions, this power system is known as the grid and can be broadly divided into the generators that supply the power (also called generation network), the transmission system that carries the power from the generating centres to the load centres (also called transmission network), and the distribution system that feeds the power to nearby homes and industries (also called distribution network). In this chapter, we start by providing a description of the physical foundations of electricity. We next discuss the structure and components of the electric power system. **Readers familiar with these concepts can skip to the next chapter.**

1.1 Fundamentals of electric power

To understand electric power systems, it is helpful to have a basic understanding of the fundamentals of electricity. These include the concepts of voltage, current, impedance, power, Ohm's law, and Kirchhoff's laws, which relate these quantities in an electric power systems.

1.1.1 Voltage

Voltage is defined as the amount of potential energy between two points on a circuit. One point has more charge than another. This difference in charge between the two points is called voltage. Voltage is measured in volts (V), and for large values (very typical for power networks) expressed in kilovolts (kV) or megavolts (MV).

1.1.2 Current

Current is a measure of the rate of flow of charge through a conductor. It is measured in amperes.

1.1.3 DC and AC

Current can be unidirectional, referred to as direct current, or it can periodically reverse directions with time, in which case it is called alternating current. Voltage also can be unipolar or alternating in polarity with time. Unipolar voltage is referred to as dc voltage. Voltage that reverses polarity in a periodic fashion is referred to as ac voltage. Alternating currents and voltages in power systems have nearly sinusoidal profiles.

AC voltage and current waveforms are defined by three parameters: amplitude, frequency, and phase. The maximum value of the waveform is referred to as its amplitude. Frequency is the rate at which current and voltage in the system oscillate, or reverse direction and return. Frequency is measured in cycles per second, also called hertz (Hz). DC can be considered a special case of AC, one with frequency equal to zero. Finally, the time in seconds it takes for an AC waveform to complete one cycle, the inverse of frequency, is called the period. The phase of an AC waveform is a measure of when the waveform crosses zero relative to some established time reference. Phase is expressed as a fraction of the AC cycle and measured in degrees (ranging from -180 to +180 degrees). There is no concept of phase in a DC system. Note that the electric power systems are predominantly AC, although a few select sections are DC.

1.1.4 Phasors

In order to simplify calculations for AC, sinusoidal voltage and current waves are commonly represented as complex-valued functions of time denoted as V and I

$$V = |V|e^{j(\omega t+\phi_V)} \tag{1.1}$$
$$I = |I|e^{j(\omega t+\phi_I)}, \tag{1.2}$$

where $|.|$ represents the amplitude, ω represents the frequency, and $\phi_{(.)}$ represents the phase of the sinusoidal wave. Since phase contribution by ωt is same for both currents and voltages, we can simply keep track of the magnitude and the phase of voltage and current. This representation is called phasor represnetation, i.e.,

$$V = |V|\angle\phi_V \tag{1.3}$$
$$I = |I|\angle\phi_I, \tag{1.4}$$

1.1.5 Impedance

Impedance is a property of a conducting device, for example, a transmission line that represents the impediment it poses to the flow of current through it when a voltage is applied. The rate at which energy flows through a transmission line is limited by the line's impedance. Impedance has two components: resistance and reactance. Impedance, resistance, and reactance are all measured in ohms.

Resistance

Resistance is the property of a conducting device to resist the flow of current through it. Resistance causes energy loss in the conductor as moving charges collide with the conductor's atoms and results in electrical energy being converted into heat.

Reactance

Voltages and currents create electric and magnetic fields, respectively, in which energy is stored. Reactance is a measure of the impediment to the flow of power caused by the creation of these fields.

Under the complex notation introduced earlier, impedance can be written as $Z := R + jX$, where the real part of impedance is the resistance R and the imaginary part is the reactance X. Above notation indicates that one can think of impedance as a generalized resistance, and can treat it in the same fashion as pure resistance, inlcuding parallel and series connection rules. Also note that j is the imaginary unit, and is used instead of i in this context to avoid confusion with the symbol for electric current.

1.1.6 Ohm's law

Ohm's law states that the current through a conductor between two points is directly proportional to the voltage across the two points, and the constant of proportionality is the impedance of the conductor. Mathematically,

$$V = IZ = I|Z|e^{j\angle(Z)} \tag{1.5}$$

1.1.7 Power

Power in an electric circuit is the rate of energy consumption or production as currents flow through various parts comprising the circuit. In alternating current circuits, immediate power transferred through any phase varies periodically. Energy storage elements such as inductors and capacitors may result in periodic reversals of the direction of energy flow. The portion of power that, averaged over a complete cycle of the AC waveform, results in net transfer of energy in one direction is known as active power (sometimes also called real power). The portion of power due to stored energy, which returns to the source in each cycle, is known as reactive power. Active power is the power that actually does work. It is measured in watts. Reactive power is measured in volt-amperes reactive (VAR).

In a simple alternating current (AC) circuit consisting of a source and a linear load, both the current and voltage are sinusoidal. If the load is purely resistive, the two quantities reverse their polarity at the same time. At every instant the product of voltage and current is positive or zero, with the result that the direction of energy flow does not reverse. In this case, only active power is transferred.

If the loads are purely reactive, then the voltage and current are 90 degrees out of phase. For half of each cycle, the product of voltage and current is positive, but on the other half of the cycle, the product is negative, indicating that on average, exactly as much energy flows toward the load as flows back. There is no net energy flow over one cycle. In this case, only reactive power flows–there is no net transfer of energy to the load.

Practical loads have resistance, inductance, and capacitance, so both active and reactive power will flow to real loads. Mathematically, we define power as a complex number too for the ease of calculation. In particular, $S := P + jQ = VI^*$, where P is active power, Q is reactive power, S is complex power, and I^* denotes the complex conjugate of current. Using the Ohm's law, it is easy to see that:

$$S = VI^* = \frac{|V|^2}{Z} = |I|^2 Z = |I|^2 R + j|I|^2 X$$

1.1.8 Kirchhoff laws

Kirchhoff's circuit laws are two equalities that deal with the current and voltage in the electrical circuits. First law (also known as KCL) states that at any node (junction) in an electrical circuit, the sum of currents flowing into that node is equal to the sum of currents flowing out of that node or equivalently the algebraic sum of currents in a network of conductors meeting at a point is zero. The second law (also known as KVL) states that the directed sum of the voltage around any closed network is zero. KCL is particularly useful in deriving the power flow equations for a power system, as we will see in the next chapter.

1.2 Structure of a power system

The electric power system consists of generating units where primary energy is converted into electric power, transmission and distribution networks that transport this power, and consumer's equipment (also called loads) where power is used. While originally generation, transport, and consumption of electric power were local to relatively small geographic regions, today these regional systems are connected together by high-voltage transmission lines to form highly interconnected and complex systems that span wide areas. We discuss each of these subsystems next.

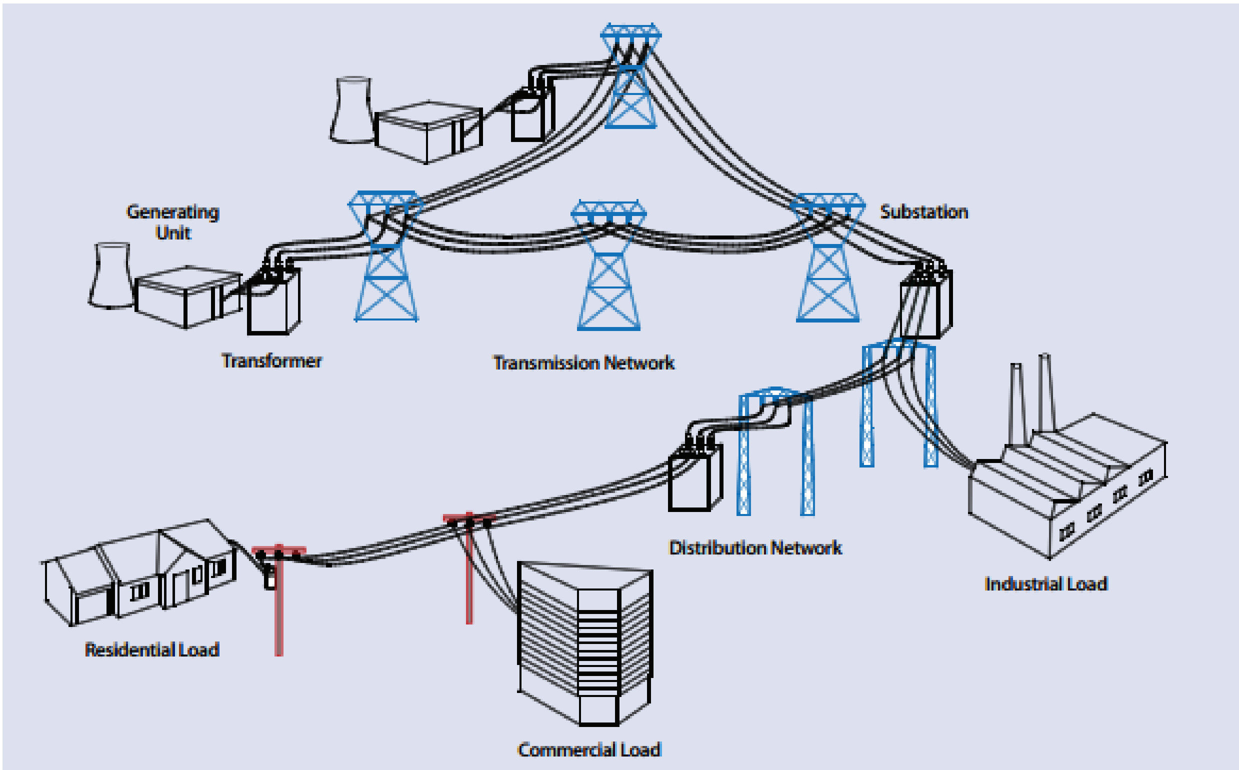


Figure 1.1: Structure of the Electric Power System

1.2.1 Generation

Electric power is produced by generating units, housed in power plants, which convert primary energy into electric energy. Primary energy comes from a number of sources, such as fossil fuel and nuclear, hydro, wind, and solar power. The process used to convert this energy into electric energy depends on the design of the generating unit, which is partly dictated by the source of primary energy. Large generating units generally are located outside densely populated areas, and the power they produce has to be transported to load centers. They produce three-phase ac voltage at the level of a few to a few tens of kV.

1.2.2 Transmission

The transmission system carries electric power over long distances from the generating units to the distribution system. The transmission network is composed of power lines and stations/substations. Topologically, the transmission line configurations are mesh (or grid) networks (as opposed to the distribution networks which are radial), meaning there are multiple paths between any two points on the network. This redundancy allows the system to provide power to the loads even when a transmission line or a generating unit goes offline. Because of these multiple routes, however, the power flow path cannot be specified at will. Instead power flows along all paths from the generating unit to the load. The power flow through a particular transmission line depends on the line's impedance and the amplitude and phase of the voltages at its ends. Predicting these flows requires substantial computing power and precise knowledge of network voltages and impedances, which are rarely known with high precision. Hence, precise prediction of the power flowing down a particular transmission line is difficult in general.

1.2.3 Distribution

Distribution networks carry power the last few miles from transmission network to consumers. Power is carried in distribution networks through wires either on poles or, in many urban areas, underground. Distribution networks are distinguished from transmission networks by their voltage level and topology. Lower voltages are used in distribution networks; typically lines up to 35 kV are considered part of the distribution network. Primary distribution lines are also called feeders. Distribution networks usually have a radial trre-like topology, so they are also referred to as a star network, radial network, or a tree network, with only one power flow path between the distribution substation and a particular load.

1.2.4 Consumption

Electricity supplied by the distribution network is finally consumed by a wide variety of loads, including lights, heaters, electronic equipment, household appliances, and motors that drive fans, pumps, and compressors. These loads can be classified based on their impedance, which can be resistive, reactive, or a combination of the two.

1.3 Components of a power systen

1.3.1 Load

Loads can be classified based on their impedance, which can be resistive, reactive, or a combination of the two. In theory, loads can be purely reactive, and their reactance can be either inductive or capacitive. Purely resistive loads only consume real power. Loads with inductive or capacitive impedance also draw reactive power. Because of the abundance of motors connected to the network, the power system is dominated by reactive loads. Hence, generating units have to supply both real and reactive power. From the power systems operational perspective, the aggregate power demand of the loads at a partiocular node/bus is more important than the power consumption of individual loads. This aggregate load profile is continuously varying. A typical load profile on an average day during summers and winters is shown in Figure 1.2. It is more expensive to meet the load demand near the peak of the load curve, as a higher generation capacity to meet the peak load is needed.

1.3.2 Local generators

In addition to the large generating units, discussed in the previous section, the power system typically also has some distributed generation. These could be local renewable energy generators, like solar power plants, wind turbines, photo-voltaic systems, etc. Such generating units are also called renewables or local generators in the power system literature. These units generally operate at lower voltages and are connected at the distribution system level. Small generating units, such as solar photovoltaic arrays, may be single-phase.

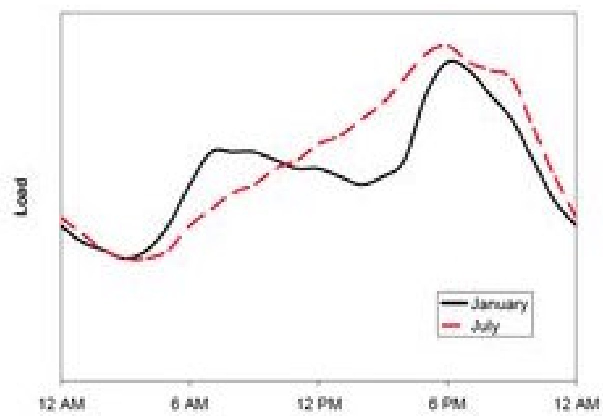


Figure 1.2: A typical load profile during summers and winters

1.3.3 Bus

In power engineering, any graph node of the power distribution network at which voltage, current, power flow, or other quantities are to be evaluated is known as bus. Each bus (or node) j is characterized by two complex variables V_j and S_j, or equivalently, four real variables. The buses are usually classified into three types, depending on which two of the four real variables are specified. For the slack bus 0, that is the main supplier of energy in a network, V_0 is assumed to be fixed and given, and S_0 is variable. For a generator bus (also called PV-bus), $Re(S_j) = P_j$ and $|V_j|$ are specified and $Im(S_j) = Q_j$ and $\angle V_j$ are variable. For a load bus (also called PQ-bus), S_j is specified and V_j is variable. Also note that as per energy conservation principle, the power flow into and out of each of the buses that are network terminals is the sum of power flows of all of the lines connected to that bus.

1.3.4 Transmission line/ Line

Transmission line is a cable that is used in electric power transmission and distribution networks to transmit electrical energy from one node (or bus) to other. Transmission lines have some impedance, so electricty transmission through these lines causes some power loss. Impedance of transmission line is sometimes also called line impedance.

Chapter 2

Power flow equations

As discussed in the previous chapter, a power network consists of several buses connected together through power lines. The equations that govern the flow of power through these lines, as well as corresponding bus voltages and line currents can be obtained by using the Kirchhoff's laws and the Ohm's law at every bus, and are called Power flow (PF) equations. However, due to the non-linear and non-convex nature of these equations, solving them to get the operating point of the network is generally very hard. To overcome this problem, some approximations and relaxations have been proposed in the literature. Since transmission and distribution networks have different structures, these relaxations and approximations are different for the two networks. In this project, we will focus on the distribution networks. We will begin with introducing a network model for the distribution networks, which have a radial tree-like structure, derive its power flow equations, and then study the linear approximation of those equations. Similar results exist for transmission networks as well. Interested reader can refer to [4] for more details.

2.1 Network Model

In a radial distribution network, each bus has exactly one parent bus. Electricity to this entire distribution network is supplied by a distribution substation, which defines the root of this radial network. The bus corresponding to the substation is calles substation bus. In this section, we first setup the network model for a radial distribution network and then characterize the power flow in such a network. To represent the network we use the notation introduced in [2] (restated here in Table 2.1 for completeness), along with some other parameters.

2.2 Power Flow Equations

To characterize the power flow in a radial distribution system, DistFlow equations are used (first introduced in [1]). These DistFlow equations can be used to find the operating point of the network using Newton-Raphson method and shown to have nice convergence properties [1]. We first consider a special case where the distribution network is a line network (linear feeder), and derive power flow (DistFlow) equations for it. General radial distribution networks are considered next.

Table 2.1: Variables for a radial distribution network

$\mathscr{N}$	Set of buses, $\mathscr{N} := \{1,\ldots,n\}$
$\mathscr{L}$	Set of lines between the buses in $\mathscr{N}$
$\mathscr{L}_i$	Set of lines on the path from bus 0 to bus i
p_i^l	Real net power consumption by load at bus i
q_i^l	Reactive net power consumption at bus i
z_{ij}	Impedence of line $(i,j) \in \mathscr{L}$
r_{ij}, x_{ij}	Resistance and reactance of line $(i,j) \in \mathscr{L}$
S_{ij}	Complex power flow from bus i to j
P_{ij}, Q_{ij}	Real and reactive power flows from bus i to j
V_i	Voltage (complex) at bus i
v_i	Magnitude of voltage at bus i
I_{ij}	Current (complex) from bus i to j
l_{ij}	Squared magnitude of complex current from bus i to j

2.2.1 Linear Feeder

We can write the power flow equations for the line network (see figure 2.1) in a recursive fashion using the Kirchhoff's laws and the Ohm's law at every bus.

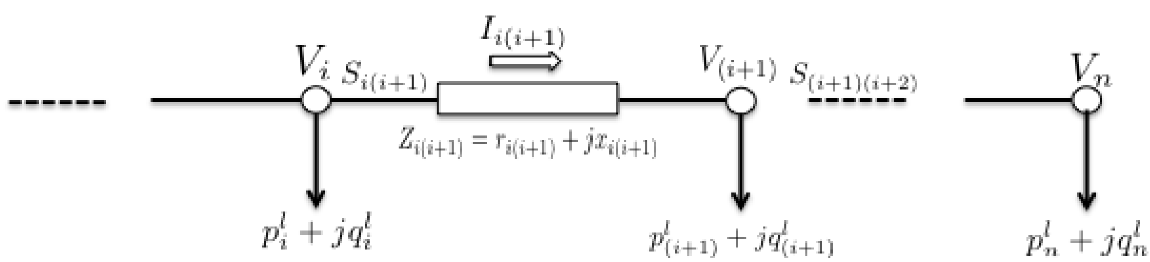


Figure 2.1: Line diagram of a linear feeder distribution network

From the conservation of power, power going out of node $i-1$ is equal to the sum of the loss in power during transmission from node $i-1$ to node i, power consumed at node i, and power going out of node i. Mathematically, this gives us the following equation (the following equations hold for all time instants so we are dropping the time index t in all equations):

$$S_{(i-1)i} = S_{i(i+1)} + l_{(i-1)i} z_{(i-1)i} + p_i^l + jq_i^l \tag{2.1}$$

Using Ohm's law across the impedence of the line connecting node $i-1$ and i:

$$I_{(i-1)i} = \frac{V_{i-1} - V_i}{z_{(i-1)i}}$$

Multiplying both sides by $z_{(i-1)i}$ and taking the magnitud, we have

$$v_i = |V_{i-1} - I_{(i-1)i} . z_{(i-1)i}| \tag{2.2a}$$

$$v_i^2 = v_{i-1}^2 + |I_{(i-1)i}.z_{(i-1)i}|^2 - 2Re\left(V_{i-1}.I_{(i-1)i}^* z_{(i-1)i}^*\right) \tag{2.2b}$$

By the definition of complex power, we have:

$$S_{(i-1)i} = P_{(i-1)i} + jQ_{(i-1)i} = V_{i-1}.I_{(i-1)i}^* \tag{2.3}$$

Combining equations (2.1), (2.2), and (2.3) will give us the recursive power flow equations for a linear feeder:

$$P_{(i-1)i} = p_i^l + r_{(i-1)i} l_{(i-1)i} + P_{i(i+1)} \tag{2.4a}$$

$$Q_{(i-1)i} = q_i^l + x_{(i-1)i} l_{(i-1)i} + Q_{i(i+1)} \tag{2.4b}$$

$$v_i^2 = v_{i-1}^2 + \left(r_{(i-1)i}^2 + x_{(i-1)i}^2\right) l_{(i-1)i} - 2\left(r_{(i-1)i} P_{(i-1)i} + x_{(i-1)i} Q_{(i-1)i}\right) \tag{2.4c}$$

$$l_{(i-1)i} v_{i-1}^2 = P_{(i-1)i}^2 + Q_{(i-1)i}^2 \tag{2.4d}$$

As stated in Table 2.1, $P_{(i-1)i}$ and $Q_{(i-1)i}$ are the real and reactive power from bus $i-1$ to bus i respectively. Similarly, v_i and $l_{(i-1)i}$ denote the voltage magnitude at bus i and squared magnitude of the current flowing from bus $i-1$ to bus i respectively. We also have the following boundary conditions:

- Substation bus voltage is assumed to be fixed and known at all times, and is normalized to 1:

$$V_0 = 1\angle 0$$

- There is no power flow at the end of the main feeder:

$$P_{n(n+1)} = Q_{n(n+1)} = 0$$

Using these boundary conditions, one can solve the power flow equations (2.4) (in theory), and can get the power flow and currents in each line, and all bus voltages; however, as stated earlier, these equations are generally hard to solve due to their non-linear nature.

2.2.2 General radial network

For a general radial network, the DistFlow equations can be easily derived by noting that instead of only one line, several lines might now be connected to a bus, and hence we have to sum the losses and power flow through each of the lines in (2.1). The rest of the derivation can be carried out in a similar fashion to get the following power flow equations:

$$P_{ij} = p_j^l + r_{ij} l_{ij} + \sum_{k:(j,k)\in\mathscr{L}} P_{jk} \tag{2.5a}$$

$$Q_{ij} = q_j^l + x_{ij} l_{ij} + \sum_{k:(j,k)\in\mathscr{L}} Q_{jk} \tag{2.5b}$$

$$v_j^2 = v_i^2 + (r_{ij}^2 + x_{ij}^2) l_{ij} - 2(r_{ij} P_{ij} + x_{ij} Q_{ij}) \tag{2.5c}$$

$$l_{ij} v_i^2 = P_{ij}^2 + Q_{ij}^2, \tag{2.5d}$$

and the following boundary conditions:

- The substation voltage is known at all times:

$$V_0 = 1\angle 0$$

- Here we do not need explicitly need a second boundary condition unlike single feeder case as our notation in equation (2.5) (i.e. using $\mathscr{L}$) already takes care of that (because the last bus has no neighbors in a lateral and hence the summation term in the equation (2.5) will be zero)

Given the load profile at every time instant, i.e. if p_j^l and q_j^l are known, one can ideally solve the above PF equations; however, that is seldom the case in reality. To overcome this problem a linear approximation of the above PF equations has been proposed in the power systems literature that can be solved to get approximate estimates of voltages and currents.

2.2.3 Linear approximation of power flow equations

We next want to reformulate the power flow equations (2.5) that are more suitable for the stability analysis and optimization. Following [2] we assume losses are small as compared to the load power i.e. $l_{ij}r_i, l_{ij}x_i \approx 0$ for all $(i,j) \in \mathscr{L}$ in (2.5). This approximation neglects the higher order real and reactive power loss terms that are generally much smaller than power flows P_{ij} and Q_{ij}, and only introduces a small relative error, typically on the order of 1% [2]. With above approximation these equations reduce to:

$$P_{ij} = p_j^l + \sum_{k:(j,k)\in\mathscr{L}} P_{jk} \tag{2.6a}$$

$$Q_{ij} = q_j^l + \sum_{k:(j,k)\in\mathscr{L}} Q_{jk} \tag{2.6b}$$

$$v_j^2 = v_i^2 - 2(r_{ij}P_{ij} + x_{ij}Q_{ij}) \tag{2.6c}$$

As in [2], we further assume that $v_i \approx 1$ (in per unit). This approximation means that the voltage drop is negligible along the network and hence all the buses can be assumed to be roughly at 1 per unit (i.e. the voltage level of the substation bus). With this approximation, voltage equation in (2.6) further simplifies to:

$$v_i - v_j = (r_{ij}P_{ij} + x_{ij}Q_{ij}) \tag{2.7}$$

as $(v_j + v_i) \approx 2$. We can now recursively write the voltage equation (2.7) to get the following vector linear equation (see [2] for more details on this derivation):

$$v = \bar{v}_0 - Rp^l - Xq^l \tag{2.8}$$

where,

$$R_{ij} = \sum_{(h,k)\in\mathscr{L}_i\cap\mathscr{L}_j} r_{hk} \tag{2.9a}$$

$$X_{ij} = \sum_{(h,k)\in\mathscr{L}_i\cap\mathscr{L}_j} x_{hk} \tag{2.9b}$$

and $\bar{v}_0 := (v_0, \ldots, v_0) \in \mathbb{R}^n$ and v_0 is voltage at the substation bus (generally assumed to be 1pu).

Although the linear PF equations are very easy to solve, their solution might not satisfy the original power flow equations. This can be a challenge, especially when the power flow equations are constraints in a broader optimization problem over the power network (see the next chapter on Optimal power flow problems for examples). In this case one relies on heuristics that are hard to scale to larger systems. To overcome this problem, convex relaxations of the power flow problems have been proposed. Next chapter will focus on one such (an SOCP) relaxation of the power flow equations.

Chapter 3

Optimal power flow problem and its relaxations

3.1 Optimal power flow problem

3.1.1 Motivation

The optimal power flow (OPF) problem determines a network operating point that minimizes a certain objective such as generation cost or power loss. It has been one of the fundamental problems in power system operation since 1962. As distributed generation (e.g., photovoltaic panels) and controllable loads (e.g., electric vehicles) proliferate, OPF problems for distribution networks become increasingly important [3]. To use controllable loads to integrate volatile renewable generation, solving the OPF problem in real-time will be inevitable.

3.1.2 Standard form of OPF problem

In a standard OPF problem, we construct an optimization problem with an objective to minimize, like the total power injections into the network, while having the voltage, current, and power flows satisfying the power flow equations for the radial network derived in the previous chapter. We inherit the notations from the previous section. Let's remind ourselves that P_{ij} and Q_{ij} is the real and reactive power flow from bus i to bus j respectively, and l_{ij} is the square of the magnitude of the current from bus i to bus j. Additionally, we introduce the notation $v_i^{'} = v_i^2$, i.e., $v_i^{'}$ is the square of the magnitude of the voltage at bus i. With this in mind, we have the standard form of the OPF problem as follows:

$$\begin{aligned} \text{minimize} \quad & f \\ \text{subject to} \quad & P_{ij} = p_j^l + r_{ij}l_{ij} + \sum_{k:(j,k)\in\mathscr{L}} P_{jk} \\ & Q_{ij} = q_j^l + x_{ij}l_{ij} + \sum_{k:(j,k)\in\mathscr{L}} Q_{jk} \\ & v_j^{'} = v_i^{'} + (r_{ij}^2 + x_{ij}^2)l_{ij} - 2(r_{ij}P_{ij} + x_{ij}Q_{ij}) \\ & l_{ij}v_i^{'} = P_{ij}^2 + Q_{ij}^2, \end{aligned} \tag{3.1}$$

Voltage constraints

In addition to the PF constraints, we also often have the voltage stability constraint. This constraint arises due to the reactance of a transmission line that causes the voltage at the far end of the line to drop below an allowable level (typically 95% of the nominal design voltage level) when the power flowing through the line exceeds a certain level. In particular, if the reactive power consumed by the load increases without a commensurate increase in reactive power supply, the output voltage of the generator will decrease. Conversely, the output voltage of the generator will increase if the generator is supplying more reactive power than is being drawn. In mathematical form, we have

$$v_{min} \leq v \leq v_{max} \tag{3.2}$$

where v_{min} and v_{max} are the minimum and maximum magnitude of voltage allowed.

Power injection constraints

Another constraint that is usually present in the OPF problem is the limit on the amount of power injection into each bus. This constraint naturally arises from the physical limitations of the amount of power that can be generated or stored at each bus. For example, if we have more power at a given node that is needed to support the downstream transmission and loads, then some power could be saved at that node. On the contrary, if there isn't enough power to support the consumption of power by the downstream network, power could be injected at that node to eliminate the deficiency. Let p_i^g, q_i^g denotes the real and complex injection power at bus i respectively. Then, in mathematical form, we have

$$\begin{aligned} p^g_{min} \leq & p^g \leq p^g_{max} \\ q^g_{min} \leq & q^g \leq q^g_{max} \end{aligned} \tag{3.3}$$

where p^g_{min} and p^g_{max} are the minimum and maximum real injection power allowed allowed, and q^g_{min} and q^g_{max} are the minimum and maximum of reactive injection power allowed.

3.1.3 Difficulty in exactly solving OPF problem

The OPF problem is difficult to solve in general because power flow is governed by nonlinear Kirchhoffs laws (see equation 2.5(d)). There are three ways that have been proposed to address this challenge:

- Approximation the power flow equations (as discussed in section 2.2.3)
- Solve for a local optimum of the OPF problem
- Convexify the constraints imposed by the Kirchhoff laws

In this report, we will focus on the approach 1 and 3. In section 2.2.3, we already discussed an alternative formulation of PF equations under the following assumptions:

- power losses on the lines are small,
- voltages are close to their nominal values,
- voltage angle differences between adjacent buses are small.

These assumptions lead to linearized power flow equations which are also called DC power flow equations (DC because phase angles are assumed to be same across all buses). For transmission networks, these three assumptions are reasonable and DC OPF is widely used in practice. It, however, has three limitations. First, it is not applicable for applications such as power routing and volt/var control since it assumes fixed voltage magnitudes and ignores reactive powers. Second, a solution of the DC approximation may not be feasible (may not satisfy the nonlinear power flow equations). In this case an operator typically tightens some constraints in DC OPF and solves again. This may not only reduce efficiency but also relies on heuristics that are hard to scale to larger systems or faster control in the future. Finally, DC approximation is unsuitable for distribution systems where loss is much higher than in transmission systems, voltages can fluctuate significantly, and reactive powers are used to stabilize voltages. To overcome this problem, convex relaxations of power flow equations have been proposed in the literature [4]. Solving OPF through convex relaxation offers several advantages. It provides the ability to check if a solution is globally optimal. If it is not, the solution provides a lower bound on the minimum cost and hence a bound on how far any feasible solution is from optimality. Unlike approximations, if a relaxation is infeasible, it is a certificate that the original OPF is infeasible. In this chapter, we will discuss one of such relaxations, which will convert OPF problem into a second order cone problem.

3.2 PF relaxations

3.2.1 Reformulating Power Flow Equations Using Second Order Cone Constraint

If we look at the standard form of the optimal power flow problem (equation 3.1), we see that the constraints are linear in the variables $P_{ij}, Q_{ij}, v_i^{'}, l_{ij}$ except the last equality constraint $l_{ij}v_i^{'} = P_{ij}^2 + Q_{ij}^2$. We could relax this non-convex equality constraints to a convex inequality constraint by replacing the constraint with $l_{ij}v_{ij}^{'} \geq P_{ij}^2 + Q_{ij}^2$. With this in mind, our optimal power flow problem becomes:

$$\begin{aligned}
\text{minimize} \quad & f \\
\text{subject to} \quad & P_{ij} = p_j^l + r_{ij}l_{ij} + \sum_{k:(j,k)\in\mathscr{L}} P_{jk} \\
& Q_{ij} = q_j^l + x_{ij}l_{ij} + \sum_{k:(j,k)\in\mathscr{L}} Q_{jk} \\
& v_j^{'} = v_i^{'} + (r_{ij}^2 + x_{ij}^2)l_{ij} - 2(r_{ij}P_{ij} + x_{ij}Q_{ij}) \\
& l_{ij}v_i^{'} \geq P_{ij}^2 + Q_{ij}^2,
\end{aligned} \tag{3.4}$$

Observe that the last inequality constraint can be converted to standard second order cone inequality form as follows

$$l_{ij}v_i^{'} \geq P_{ij}^2 + Q_{ij}^2 \Leftrightarrow \left\| \begin{bmatrix} 2P_{ij} \\ 2Q_{ij} \\ l_{ij} - v_i^{'} \end{bmatrix} \right\|_2 \leq l_{ij} + v_i^{'}.$$

Rewriting our relaxed OPF problem in standard SOCP form, we have

$$\begin{aligned}
\text{minimize} \quad & f \\
\text{subject to} \quad & P_{ij} = p^l_j + r_{ij}l_{ij} + \sum_{k:(j,k)\in\mathscr{L}} P_{jk} \\
& Q_{ij} = q^l_j + x_{ij}l_{ij} + \sum_{k:(j,k)\in\mathscr{L}} Q_{jk} \\
& v'_j = v'_i + (r^2_{ij} + x^2_{ij})l_{ij} - 2(r_{ij}P_{ij} + x_{ij}Q_{ij}) \\
& \left\| \begin{bmatrix} 2P_{ij} \\ 2Q_{ij} \\ l_{ij} - v'_i \end{bmatrix} \right\|_2 \leq l_{ij} + v'_i.
\end{aligned} \tag{3.5}$$

Note that SOCP is not necessarily convex, since we allow f to be nonconvex. Nonetheless, we call it SOCP for brevity.

Remark 1 *Some other relaxations have also been proposed in literature for the PF equations, for example SDP relxation coverts the OPF problem into a relaxed SDP problem, all these relaxations have been proved to be equivalent for radial networks in the sense that there is a bijective map between their feasible sets [4]. The SOCP relaxation, however, has a much lower computational complexity. We will hence focus on the SOCP relaxation in this project.*

3.2.2 Conditions for exactness

We say a convex relaxation of the power flow equations is exact if an optimal solution of the original nonconvex OPF can be recovered from every optimal solution of the relaxation. In general, the SOCP relaxation of the power flow equations is not exact; however, under some additional conditions on network parameters and voltage bounds, it can proved to be exact. These conditions are not necessary in general and have implications on allowable voltage magnitudes. In this section, we present those sufficient conditions and prove the exactness of SOCP relaxation under those conditions. In particular, we prove that if voltage upper bounds do not bind at optimality, then the SOCP relaxation is exact under a mild condition. The condition can be checked a priori and holds for a majority of the real-world systems [3]. The condition has a physical interpretation that all upstream reverse power flows increase if the power loss on a line is reduced.

We start with introducing the notations that will be used in the statement of the condition. Recall from the section 2.2.3, one can ignore the l terms to obtain the DC power flow equations:

$$S_{ij} = -s_j + \sum_{k:(j,k)\in\mathscr{L}} S_{jk} \tag{3.6a}$$

$$v^2_j - v^2_i = -2Re(z^*_{ij}S_{ij}), \tag{3.6b}$$

where $s_j = -p^l - jq^l$.

Adopting an orientation of everything directing towards substation bus (as opposed to away from the substation bus), we can rewrite the above equations for $(i, j) \in \mathscr{L}$ as:

$$\hat{S}_{ji} = s_j + \sum_{k:(j,k)\in\mathscr{L}} \hat{S}_{kj} \tag{3.7a}$$

$$\hat{v}_j^2 - \hat{v}_i^2 = 2Re(z_{ij}^* \hat{S}_{ji}), \tag{3.7b}$$

This orientation is more useful to deal with to prove the sufficient conditions for exactness. Let (S^{lin}, v^{lin}) denotes the solution to the above DC power flow equations, then

$$\hat{S}_{ji}^{lin} = \sum_{k \in \mathscr{T}_j} s_k \tag{3.8a}$$

$$\hat{v}_j^{lin^2} = v_0^2 + 2 \sum_{(m,k) \in \mathscr{L}_j} Re(z_{mk}^* \hat{S}_{km}^{lin}), \tag{3.8b}$$

where $\mathscr{T}_j$ denotes the subtree at node j, including j. From hereon, we remove the symbol $\hat{}$ for brevity.

We define the 2×2 matrix function:

$$\underline{A}_{ji} := I - \frac{2}{\underline{v}_j} z_{ij} \left(\left[S_{ji}^{lin} \right]^+ \right)^\top, \tag{3.9}$$

where $z_{ij} := [r_{ij} x_{ij}]^\top$ is the line impedance, $S_{ji} := [P_{ji} Q_{ji}]^\top$ is the branch power flow, I is 2×2 identity matrix, and a^+ is defined as $max\{a, 0\}$. Finally, since our network is radial, every line ji means that i is the unique parent of j. Therefore, we refer to ji as j in the equations below for brevity.

Theorem 1 *Assume*

- *C1: The cost function is $f(s) := \sum_{j=0}^n f_j(Res_j)$ with f_0 strictly increasing.*
- *C2: The optimal power injection of SOCP, s, satisfies $v_j^{lin}(s) < \bar{v}_j$.*
- *C3: For each leaf node $j \in \mathscr{N}$, let the unique path from j to o has k links and be denoted by $\mathscr{P}_j := ((i_k, i_{k1}), \ldots, (i_1, i_0))$ with $i_k = j$ and $i_0 = 0$. Then $\underline{A}_{i_t} \ldots \underline{A}_{i_{t'}} z_{i_{t'+1}} > 0$ for all $1 \le t \le t' < k$.*

Then SOCP relaxation of OPF is exact.

We now comment on the conditions C1C3. First note that the typical cost functions in OPF, for example line losses, real power generation satisfy C1. Theorem 1 thus implies that if C2 holds, i.e., optimal power injections lie in the region where voltage upper bounds do not bind, then SOCP is exact under C3. C2 is affine in the injections $s := (p, q)$. It enforces the upper bounds on voltage magnitudes through a linear constraint defined by DC power flow equations. C3 is a technical assumption and has a simple interpretation: the branch power flow S_{jk} on all branches should move in the same direction. Specifically, given a marginal change in the complex power on line (k, j), the 22 matrix $\underline{A}_{jk}$ is (a lower bound on) the Jacobian and describes the effect of this marginal change on the complex power on the line immediately upstream from line (k, j). The product of $\underline{A}_i$ in C3 propagates this effect upstream towards the root. C3 requires that a small change, positive or negative, in the power flow on a line affects all upstream branch powers in the same direction. This seems to hold with a significant margin in practice; see our simulations in the next chapter or [3] for examples from real systems.

We next illustrate the proof idea of Theorem via a 3-bus linear network in Fig. 3.1. The proof for general radial networks can be found in Theorem 4, [5]. Assume C1,C2 and C3 hold. If SOCP is not exact, then there exists an optimal SOCP solution $w = (s, S, v, l)$ that violates the equality in (2.5)(d). We will construct another feasible point $w' = (s', S', v', l')$ of SOCP that has a smaller objective value than w, contradicting the optimality of w and implying SOCP is exact.

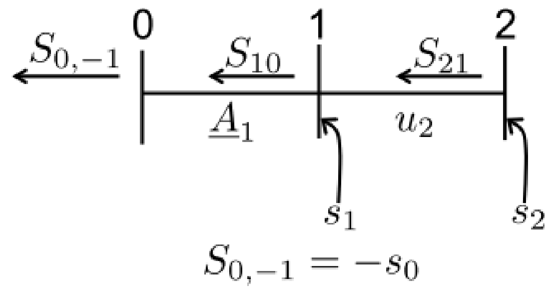


Figure 3.1: A 3-bus linear feeder

There are two ways (2.5)(d) gets violated: 1) (2.5)(d) is violated on line $(1,0)$; or 2) (2.5)(d) is satisfied on line $(1,0)$ but violated on line $(2,1)$. To illustrate the proof idea, we focus on the second case, i.e., the case where $l_{10} = \frac{|S_{10}|^2}{v_1^2}$ and $l_{21} > \frac{|S_{21}|^2}{v_2^2}$. In this case, the construction of w' is:

$$
\begin{aligned}
\text{Initialization:} s' &= s, S_{21}' = S_{21};\\
\text{Forward sweep:} l_{21}' &= |S_{21}'|^2/v_2^2,\\
S_{10}' &= S_{21}' - z_{21}l_{21}' + s_1';\\
l_{10}' &= |S_{10}'|^2/v_1^2,\\
S_{0,-1}' &= S_{10}' - z_{10}l_{10}';\\
\text{Backward sweep:} v_1' &= v_0 + 2Re(z_{10}^* S_{10}') - |z_{10}|^2 l_{10}';\\
v_2' &= v_1' + 2Re(z_{21}^* S_{21}') - |z_{21}|^2 l_{21}',
\end{aligned}
$$

where $S_{0,-1}' = -s_0'$. The construction consists of three steps: in the initialization step, s' and S_{21}' are initialized as the corresponding values in w. In the forward sweep step, $l_{k,k1}'$ and $S_{k1,k2}'$ are recursively constructed for $k = 2,1$ by alternatively applying (2.5)(d) (with v' replaced by v) and ((2.5)(a) / (2.5)(b). This recursive construction updates l' and S' alternatively along the path P2 from bus 2 to bus 0, and is therefore called a forward sweep. In the backward sweep step, v_k' is recursively constructed for $k = 1,2$ by applying (2.5)(c). This recursive construction updates v' along the negative direction of $\mathscr{P}_2$ from bus 0 to bus 2, and is therefore called a backward sweep.

One can show that w' is feasible for SOCP and has a smaller objective value than w. This contradicts the optimality of w, and therefore SOCP is exact.

Finally, to motivate our sufficient condition, we explain a simple geometric intuition using a two-bus network on why relaxing voltage upper bounds guarantees exact SOCP relaxation. Consider bus 0 and bus 1 connected by a line with impedance $z := r + ix$. Suppose without loss of generality that $v_0 = 1pu$. Eliminating $S_{01} = s_0$ from teh p[ower flow equations (2.5), the model reduces to (dropping the subscript on l_{01}):

$$p_0 - rl = -p_1, q_0 - xl = -q_1, p_0^2 + q_0^2 = l \tag{3.10}$$

and

$$v_1 - v_0 = 2(rp_0 + xq_0) - |z|^2 l \tag{3.11}$$

Suppose s_1 is given (e.g., a constant power load). Then the variables are (l, v_1, p_0, q_0) and the feasible set consists of solutions of (3.2) and (3.3) subject to additional constraints on (l, v_1, p_0, q_0). The case without any constraint is instructive and shown in Figure 3.2. The point c in the figure corresponds to a power flow solution with a large v_1 (normal operation) whereas the other intersection corresponds to a solution with a small v_1 (fault condition). As explained in the caption, SOCP relaxation is exact if there is no voltage constraint and as long as constraints on (l, p_0, q_0) does not remove the high-voltage (normal) power flow solution c. Only when the system is stressed to a point where the high-voltage solution becomes infeasible will relaxation lose exactness. This agrees with conventional wisdom that power systems under normal operations are well behaved.

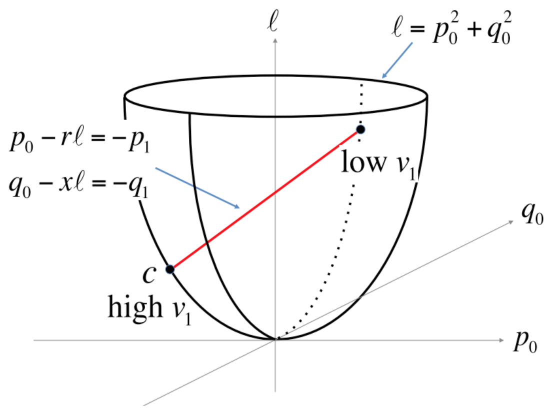


Figure 3.2: Feasible set of OPF for a two-bus network without any constraint. It consists of the (two) points of intersection of the line with the convex surface (without the interior), and hence is nonconvex. SOCP relaxation includes the interior of the convex surface and enlarges the feasible set to the line segment joining these two points. If the cost function f is increasing in l or (p_0, q_0) then the optimal point over the SOCP feasible set (line segment) is the lower feasible point c, and hence the relaxation is exact. No constraint on l or (p_0, q_0) will destroy exactness as long as the resulting feasible set contains c.

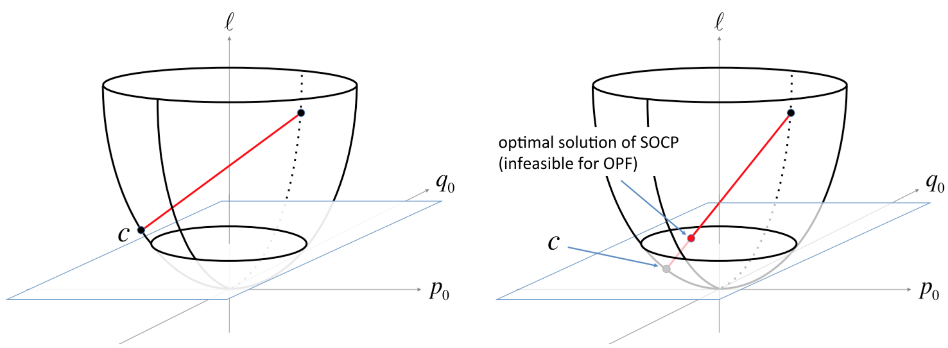


Figure 3.3: Impact of voltage upper bound $\bar{v}_1$ on exactness. (a) When $\bar{v}_1$ (corresponding to a lower bound on l) is not binding, the power flow solution c is in the feasible set of SOCP and hence the relaxation is exact. (b) When $\bar{v}_1$ excludes c from the feasible set of SOCP, the optimal solution is infeasible for OPF and the relaxation is not exact.

Consider now the voltage constraint $\underline{v}_1 \le v_1 \le \bar{v}_1$. Substituting (3.2) into (3.3) we obtain

$$v_1 = (1 + rp_1 + xq_1)|z|^2 l \tag{3.12}$$

translating the constraint on v_1 into a box constraint on l:

$$\frac{1}{|z|^2}(rp_1 + xq_1 + 1\bar{v}_1) \leq l \leq \frac{1}{|z|^2}(rp_1 + xq_1 + 1\underline{v}_1)$$

Figure 3.2 shows that the lower bound $\underline{v}_1$ (corresponding to an upper bound on l) does not affect the exactness of SOCP relaxation. The effect of upper bound $\bar{v}_1$ (corresponding to a lower bound on l) is illustrated in Figure 3.3. As explained in the caption of the figure SOCP relaxation is exact if the upper bound $\bar{v}_1$ does not exclude the high-voltage power flow solution c and is not exact otherwise.

Chapter 4

Simulations

In this chapter, we will perform some numerical simulations on some standard IEEE power systems to compare how relaxation and linear approximation perform compare to the exact solution. We start with noting that the convex relaxation need not always be exact, and the examples for the same exist in the power system literature. For example, [6] discusses a simple 2-bus example where the SOCP relaxation is not exact. However, the upper voltage bound used to construct this example is $+2\%$, which is way lower than the typical allowed limits of $5-10\%$ (and hence the upper voltage bounds are binding and the SOCP relaxation is more likely to fail to find an exact solution by virtue of Theorem 1). Our simulations illustrate that in real scenarios, however, it is fairly hard to generate conditions under which the SOCP relaxation is not exact.

4.1 Simulation settings

In this section we will discuss the basic simulation settings like network structure, load profile, etc.

4.1.1 Phases

In a typical real world setting, the power generation has three phases. In our simulation, we assume the three phases are balanced with respect to each other. In this case, one can decouple the three phases and solve them separately.

4.1.2 Exact solution

In order to compare our approximation and relaxation method with the exact solution, we used sequential convex programming to solve for the exact solution for the nonconvex optimal power flow problem. SCP in general is not guaranteed to provide the global optimum, but it is widely used in literature as a tractable way to find a solution to OPF problem.

4.1.3 Network structure

In our experiments, we are using a real 47-bus SCE distribution system [6], as depicted in the Figure 4.1. We model the closed circuit switches as shorted lines and ignore open circuit switches.

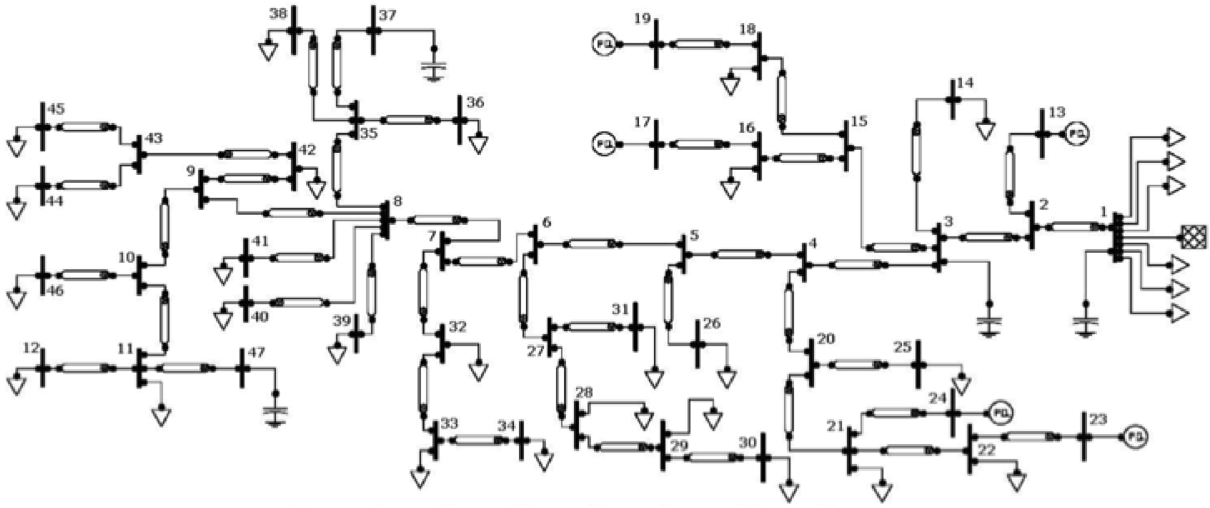


Figure 4.1: Structure of the SCE 47-bus network

4.1.4 Load profile

Each of the 47 buses in the network has a real and reactive load. We generate the load profile over 30 and 9 time steps, respectively, in the first and second example. For each bus, the load is monotonically increasing in the first half time steps, and the load for the second half of the timesteps is the flipped version of the first half time steps. This is consistent with the symmetrical load profile assumption generally used in the power system literature. A load profile example has been depicted in figure 4.2.

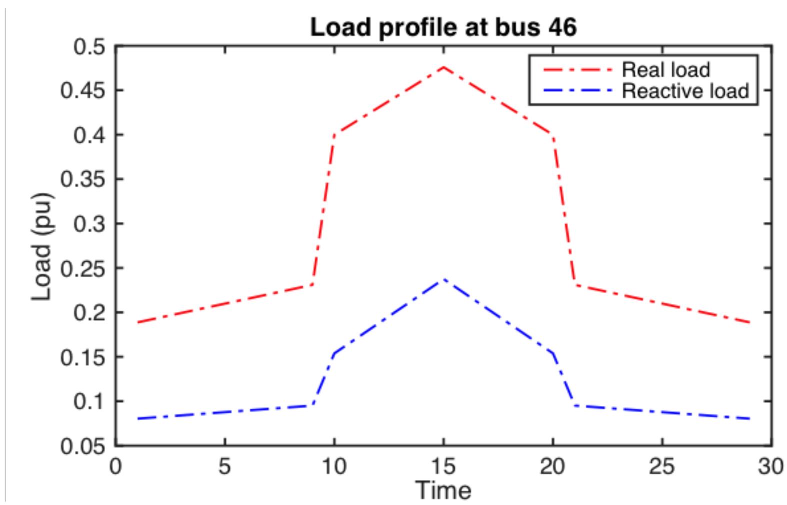


Figure 4.2: Load profile for a typical bus

4.1.5 OPF problem

In our experiments, the optimal power flow problem we were interested in is minimizing the total reactive power injection into the system, subject to the power flow equations and constraints on the voltage and injection capacity. We use the notation q_i^g to denote the injection power at bus i. Mathematically, our optimization problem is

$$
\begin{aligned}
\underset{P_{ij},Q_{ij},v_i',l_{ij},q_i^g}{minimize} \quad & \sum q_i^g \\
\text{subject to} \quad & P_{ij} = p_j^l + r_{ij}l_{ij} + \sum_{k:(j,k)\in\mathscr{L}} P_{jk} \\
& Q_{ij} = (q_j^l - q_j^g) + x_{ij}l_{ij} + \sum_{k:(j,k)\in\mathscr{L}} Q_{jk} \\
& v_j^{'} = v_i^{'} + (r_{ij}^2 + x_{ij}^2)l_{ij} - 2(r_{ij}P_{ij} + x_{ij}Q_{ij}) \\
& l_{ij}v_i^{'} = P_{ij}^2 + Q_{ij}^2 \\
& v_{min}^2 \le v' \le v_{max}^2 \\
& q_{min}^g \le q^g \le q_{max}^g
\end{aligned}
\tag{4.1}
$$

4.2 Example 1

In this example, we simulate the voltage profiles and optimal cost over time obtained via DC OPF, SOCP OPF and the actual OPF. We use a voltage lower bound of $0.9pu$ and an upper bound of $1.1pu$. Maximum capacity bound for the reactive power generator is set equal to the reactive load at that bus. It is proved in Proposition 2, [3] that if the net real and reactive power is consumed (as opposed to supplied) at every bus in the network, then all conditions of Theorem 1 are satisfied, and hence the relaxation is exact. Therefore, we should expect the SOCP solution to coincide with the true solution.

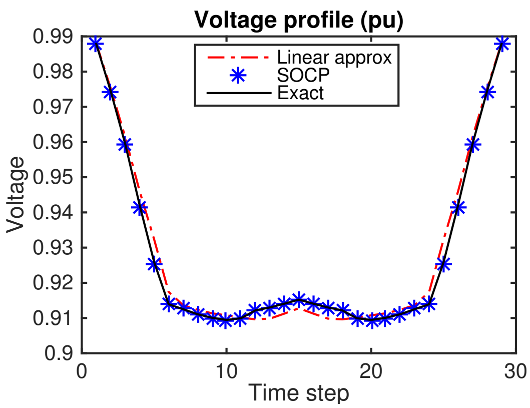
The simulated voltage profiles for the three cases for bus 20 and 46 are illustarted in the Figures 4.3a and 4.3b, respectively. As expected, the solution of SOCP relxation coincides with that of the actual OPF, and hence the relaxation is exact. This is also evident from the optimal cost curves (over time) for SOCP relaxation and actual OPF (Figure 4.5), which perfectly coincide with each other.

Interestingly enough, the voltage profile obatined via a linear approximation is very close to the true voltage profile, the assumption used to derive the linear approximation (having voltages around $1pu$), clearly, does not hold. Finally, note that the optimal generation profile obtained via linear approximation is very different from the actual optimal profile, as illustrated in the Figures 4.4a and 4.4b, indicating that a linear approximation is not suitable for designing a power generation controller.

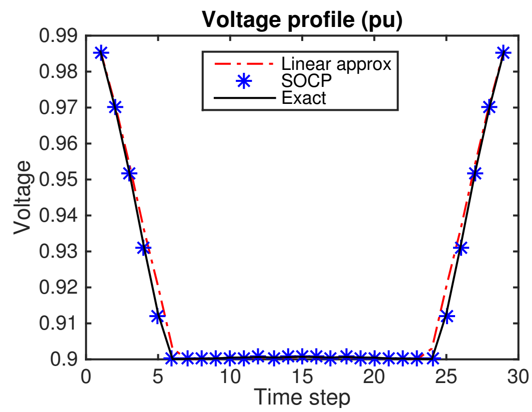
4.3 Example 2

In this example, we empirically show that it is fairly hard to render the SOCP relaxation non-exact for real-life systems. We again consider the SCE 47-bus system. We use a voltage lower bound of $0.9pu$ and an upper bound of $1.05pu$. Maximum capacity bound for the reactive power generator is set equal to the reactive load at that bus, but now we also have an active power generator at every bus. The nominal outputs of these active power generators are set at the peak loads at the corresponding buses. We then step-by-step increase the outputs of thess generators by a factor f such that the overall real power consumption at bus i is given by $p_{net}^l(i) = p_i^l - f p_{nom}^g(i)$, where $p_{nom}^g(i)$ is the nominal output of generator at bus i. When $f = 0$, we fall back to the setting in example 1.

It is important to note that when $f > 1$, every bus in net is generating active power (as opposed to being consumed) and hence the conditions in Theorem 1 are not satisfied (as discussed in example 1). Therefore,

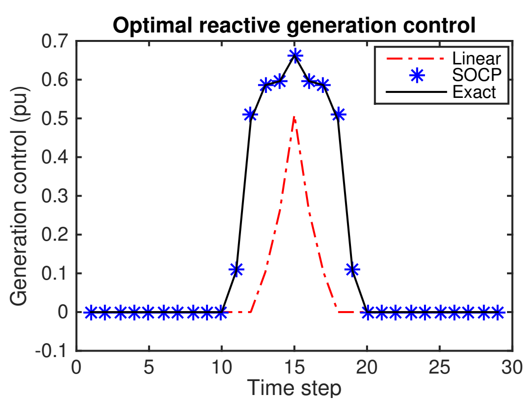


(a) Bus 20

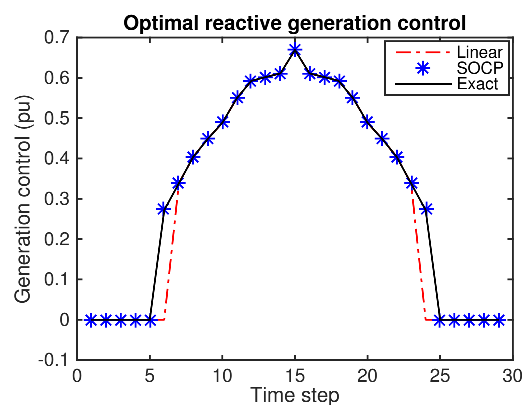


(b) Bus 46

Figure 4.3: Voltage profile at bus 20 and 46 obtained via solving DC OPF, SOCP relaxation and the exact OPF. SOCP relaxation is exact in this case.



(a) Bus 20



(b) Bus 46

Figure 4.4: Optimal generation control obtained at bus 20 and 46 via solving DC OPF, SOCP relaxation and the exact OPF. Note a significant difference between the optimal control of exact OPF and DC OPF, indicating that the linear approximation might not be suitable for generation control.

we don't necesarily expect the SOCP relaxation to be exact. However, simulation results indicate that the SOCP relaxation is actually exact for a large range of net positive active power injections, way beyond the point the conditions in Theorem 1 are violated. In particular, optimal cost curves for DC OPF, SOCP and exact OPF have been plotted for $f = 0, 2, 2.5$ and 4 in the Figure 4.6. As clear from the figure, SOCP relaxation is exact for $f = 0$ and 2, barely non-exact for 2.5 and non-exact for 4. This indicates that we can increase the generation in the grid by $200 - 250\%$ (of peak load) before the SOCP relaxation is non-exact, a state seldom achieved in any real power system. Therefore, for all practical purposes, SOCP relxation provides a tractable way to solve OPF problems.

Also note that the optimal cost for linear approximation is significantly different from the true optimal cost, and hence not suitable for the controller design. In fact, linear approximation is not feasible for 4 time points (the first two and the last two) when $f = 2, 2.5$, and is only feasible for one time point when $f = 4$. Exact OPF and SOCP OPF on the other hand are feasible for all points for these fs. This confirms that the

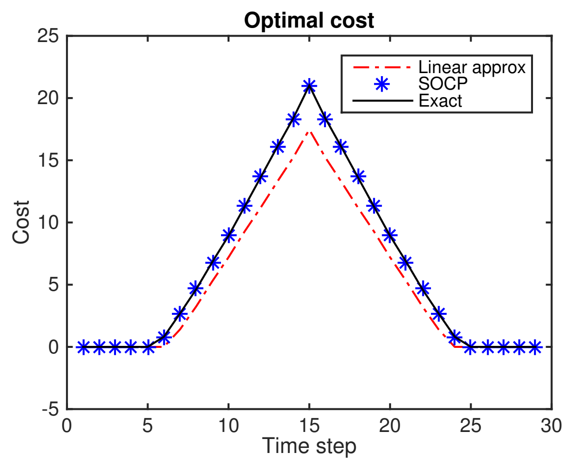
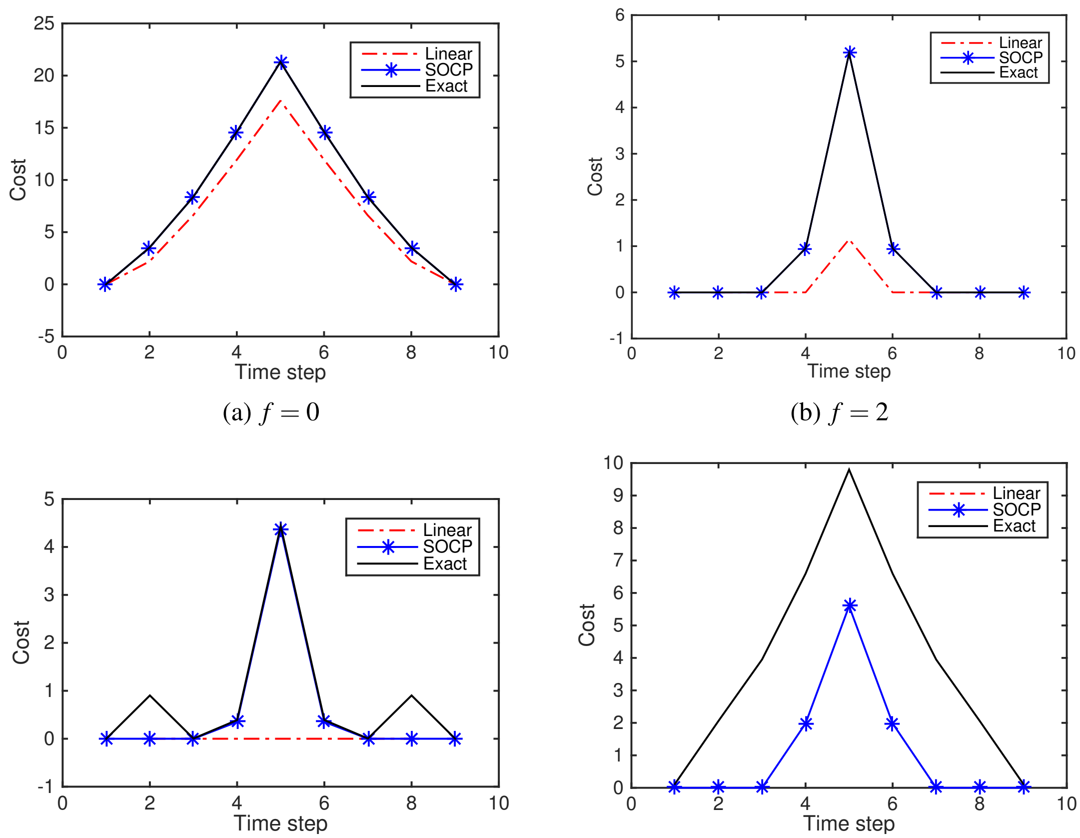


Figure 4.5: Optimal cost obtained via solving DC OPF, SOCP relaxation and the exact OPF. SOCP relaxation is exact in this case. Also note a significant difference between the optimal cost of exact OPF and DC OPF.

linear approximation doesn't provide any information about the feasibility/infeasibility of the original OPF.



(a) $f = 0$

(b) $f = 2$

(c) $f = 2.5$

(d) $f = 4$

Figure 4.6: Optimal cost obtained via solving DC OPF, SOCP relaxation and the exact OPF for different fs. SOCP relaxation is exact even when the generated power is twice as much as the peak load ($f = 2$), and barely non-exact for $f = 2.5$, indicating a high applicability of SOCP relaxation for practical examples. Linear approximation, on the other hand, is not even feasible for sereral time steps for $f = 2$, even when the exact OPF is.

Chapter 5

Conclusion

In this project, we compared two different approaches to solve OPF problems: SOCP relaxation and linear approximation. Linear approximation, in general, is unsuitable for distribution systems where loss is much higher than in transmission systems. Solving OPF through convex relaxation on the other hand provides the ability to check if a solution is globally optimal. Unlike approximations, if a relaxation is infeasible, it is a certificate that the original OPF is infeasible. We started with defining the OPF problem, and explain why they are hard to solve. We then derive the SOCP and linear versions of OPF problem, and prove that the SOCP relaxation is exact if every bus is consuming active and reactive power.

We simulated the two approaches on a real SCE 47-bus system. Our simulations confirms that the the optimal cost function obtained from linear approximation is significantly different from that of exact OPF, and hence not suitable for the controller design problem. SOCP too in general need not provide us the exact solution, but there are sufficient conditions under which we can prove there exactness. Our simulations empirically also show that the relaxation is exact beyond these conditions, indicating the wide applicability of these relaxations for real systems.

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